MAS220: Algebra
Lecturer: Dr. Neil Dummigan (Room J8, Telephone 23713)
Semester: Year
Credits: 20

The definition of abstract algebraic structures such as groups, rings, and vector spaces, is dependent on the concepts of set theory, and did not occur until the late nineteenth and early twentieth centuries. That many of the things mathematicians had already been studying turned out to be examples shows that these are the right definitions, not just exercises in playing with axioms. Through them we achieve a stunning unification of diverse areas of mathematics. The aim of this module is not only to build abstract theories, but to use them to obtain a deeper understanding of familiar mathematics, including arithmetic, coordinate geometry, vectors, calculus, linear and differential equations, with one eye on applications, to underline the significance of this mathematics.
Outline syllabus:

1. Quotient Groups
2. Conjugation in Groups
3. Group Homomorphisms
4. Introduction to Rings
5. Ring Homomorphisms
6. Divisibility and Factorisation
7. Vector Spaces
8. Linear Maps
9. Conjugation of Matrices
10. Inner Products
11. Self-adjoint Operators

Module Format:
Lectures: 43 (2 per week). Notes available at the module home page http://www.neil-dummigan.staff.shef.ac.uk/220page.html. I will elaborate on parts of these in lectures. You might like to print them out and bring them to annotate. Perhaps bring extra paper or print them out one-sided if you fear running out of space. (I have not left gaps.)
For group questions in class, we will use the L3cture app (for Android and Apple), see https://www.l3cture.com/.
Office hours: Thursday 11am, in J8 Hicks.
Tutorials: In Semester 1, Weeks 2,4,6,9,11, at whatever time and place is allocated to the group to which you are assigned. Take the problem booklet

(available at the module home page) to tutorials, since it contains all questions set in tutorials (and for homework).
Homework due at each tutorial, typically given back at the next. First homework set in Week 1 lectures, last given back in Week 12 lectures.
Though I will eventually post solutions to the problems, the intention is to learn from doing them during the semester, not to save them to the end and memorise the solutions.

Books (A=Core text, B=Secondary text, C=Background reading):
C: Jordan and Jordan, *Groups*, Modular Mathematics Series, Newnes;
C: Carter, *Visual Group Theory*, Mathematical Association of America;
C: Cameron, *Introduction to Algebra*, OUP;
C: Herstein, *Abstract Algebra*, Wiley;
C: Chatters and Hajarnavis, *An Introductory Course in Commutative Algebra*, Oxford Science Publications;
C: Allenby, *Rings, Fields and Groups*, Arnold;
C: Halmos, *Finite-Dimensional Vector Spaces*, Springer;
C: Nicholson, Linear Algebra with Applications, McGraw-Hill;
C: Lay, *Linear Algebra and its Applications*, Pearson International Edition;
C: Kaye and Wilson, *Linear Algebra*, Oxford Science Publications.

Assessment: Two short tests during lecture time, (5% each), Thursday Week 9 Semester 1 and Thursday Week 2 Semester 2. NOTE: THIS IS DIFFERENT FROM WHAT THE HANDBOOK SAID.
One formal 2.5 hour exam (90%) at the end of Semester 2. Format: All questions compulsory, length and number not fixed, total 60 marks.

Detailed Syllabus:

1. **Quotient Groups**(4 lectures)

   Groups, subgroups, isomorphisms. $\mathbb{Z}$, O_2, $\mathrm{GL}_2(\mathbb{R})$ and S_n as examples.

   Putting a group structure on the set of cosets, various incarnations of even and odd. Normal subgroups, quotient groups.

2. **Conjugation in Groups**(3 lectures)

   Conjugation as a group action, conjugacy classes as orbits. Conjugation in $\mathrm{GL}_2(\mathbb{R})$, O_2 and S_n. Normal subgroups as unions of conjugacy classes. The centre, the class equation, application to p-groups.

3. **Group Homomorphisms**(3 lectures)

   Homomorphisms, image subgroups and kernel normal subgroups. First Isomorphism Theorem for groups. Representations, Buckminsterfullerene.

4. **Introduction to Rings**(4 lectures)

   Ring axioms. Commutative and non-commutative rings, units, division rings, fields. Examples: polynomial rings, Gaussian integers, Hamilton's quaternions, matrix rings, Weyl algebra.

5. **Ring Homomorphisms**(2 lectures)

   Ring homomorphisms, inclusion and evaluation examples. First Isomorphism Theorem for rings.

6. **Divisibility and Factorisation**(6 lectures)

   Divisibility, integral domains, Euclidean domains, Euclid's algorithm. Irreducibles, associates, modular arithmetic in Euclidean domains, rings of congruence classes. Unique factorisation in Euclidean domains, application to the Gaussian integers and the two-square theorem.

7. **Vector Spaces**(5 lectures)

   Vectors in the plane and space. Cartesian coordinates. Spaces of linear functions, lines and planes and their bases. n-dimensional space, $\mathbb{R}^n$ and linear equations. F^n for any field, the ASCII code. Vector spaces. Subspaces, including null spaces and spans. Basis and dimension. $\mathbb{C}, \mathbb{F}_4$ and $\mathbb{H}$ as vector spaces. Infinite-dimensional spaces of continuous functions.

8. **Linear Maps**(7 lectures)

   Evident isomorphisms. Homomorphisms of vector spaces, example $F^n \to F$, re-name linear maps. Example $F^n \to F^m$, matrices. Linear coordinate changes, geometrical transformations. Evaluation, differentiation and integration of functions as linear maps, linear differential equations.

   Ring of linear operators, group of units, Weyl algebra revisited.

   Image and kernel subspaces for linear maps. Null spaces, column spaces, rank. Quotient spaces, restriction of functions. First Isomorphism Theorem for vector spaces. Rank-Nullity Theorem.

9. **Conjugation of Matrices**(1 lecture)

   Matrix of a linear operator with respect to a basis. Change of basis. Trace, determinant, eigenvalues and eigenvectors of a linear operator. Crystals.

10. **Inner Products**(5 lectures)

    Dot product of vectors in the plane or space, geometrical demonstration of symmetry and bilinearity. Deduction of algebraic formula, Pythagoras' Theorem. Dot product, lengths and angles (well-defined?) in $\mathbb{R}^n$. Real inner product spaces, Cauchy-Schwarz Inequality, Triangle Inequality. Substitution of integration for addition for inner products of functions. Orthogonality of trigonometric functions.

    Orthogonal complements. Linear functions, lines and planes, and Rank-Nullity, all revisited. Orthogonal projection, Gram-Schmidt process, Legendre polynomials. Fourier coefficients as inner products.

11. **Self-adjoint Operators**(3 lectures)

The adjoint property of the transpose of a matrix, general definition of the adjoint of an operator. Self-adjoint operators, real eigenvalues and orthogonal eigenvectors for distinct eigenvalues. Integration by parts, self-adjoint differential operators, orthogonality of trigonometric functions. Solution of Legendre's equation using orthogonality of Legendre polynomials. Complex inner product spaces, Spectral Theorem.

Spectral Theorem, proof for self-adjoint operators on finite-dimensional real inner product spaces.

Estimated numbers: 159