The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2014–2015**

Algebra **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Total marks 60.

1 (i) Let G be a group, $x \in G$. What is meant by the *conjugacy class* of x? ***(1 mark)***

(ii) What does it mean to say that G is abelian? ***(1 mark)***

(iii) Prove that if G is abelian then every conjugacy class has size 1. ***(1 mark)***

2 Let S_4 be the group of permutations of the set $\{1, 2, 3, 4\}$.

(i) Write down all the elements of the conjugacy class containing the element (1 2)(3 4). ***(1 mark)***

(ii) Let $V_4 = \{\text{id.}, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$. Construct a multiplication table for V_4 and use it to prove that V_4 is a subgroup of S_4. (You needn't justify each product.) Why is it a normal subgroup? What is the order of the quotient group S_4/V_4? ***(5 marks)***

(iii) Why is V_4 not isomorphic to the group $\mathbb{Z}/4\mathbb{Z}$? Is there any subgroup of S_4 isomorphic to $\mathbb{Z}/4\mathbb{Z}$? (Either write one down without justification, or give a brief reason why one cannot exist.) ***(2 marks)***

(iv) Let $a = (1\ 2)$ and $b = (1\ 2\ 3)$ in S_4. Calculate ab and ba. Hence show that the quotient group S_4/V_4 is non-abelian. ***(2 marks)***

3 (i) Let R be a ring. Let 0 be the neutral element for R as an additive group. Assuming only the ring axioms, prove that $r.0 = 0$ for all $r \in R$.
(2 marks)

(ii) Let R and S be rings, and $f : R \to S$. What does it mean to say that f is a ring homomorphism? ***(2 marks)***

(iii) If R is a commutative ring, what is meant by an *ideal* of R? If $f : R \to S$ is a homomorphism of commutative rings, define the *kernel* $\ker(f)$, and prove that it is an ideal of R. Prove also that if $\ker(f) = \{0\}$ then f is injective.
(7 marks)

4 Consider the quotient ring $R := \dfrac{\mathbb{F}_2[x]}{\langle x^3 + x + 1\rangle}$, where $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ is the field with 2 elements. Explain how R may be viewed as both a 3-dimensional vector space and a 1-dimensional vector space. ***(3 marks)***

5 Consider the matrix ring $M_2(\mathbb{R})$, with the usual addition and multiplication.

(i) Given an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $M_2(\mathbb{R})$, how do you tell whether or not it has a multiplicative inverse in $M_2(\mathbb{R})$? ***(1 mark)***

(ii) Given an invertible element $B \in M_2(\mathbb{R})$, we can define a map $f_B : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ by

$$f_B(A) := BAB^{-1} \quad \forall A \in M_2(\mathbb{R}).$$

Prove that f_B is a bijection. (You might like to consider the inverse function.) If $B = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$, calculate $f_B\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right)$. ***(3 marks)***

(iii) Prove that the map f_B, defined in the previous part, is, for any invertible $B \in M_2(\mathbb{R})$, a ring isomorphism. ***(3 marks)***

(iv) Is $B = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$ invertible as an element of the subring $M_2(\mathbb{Z})$? How about $C = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$? Find an invertible element D of $M_2(\mathbb{Z})$ whose first column is $\begin{pmatrix} 59 \\ 13 \end{pmatrix}$. ***(4 marks)***

6 (i) Let $f : \mathbb{C} \to \mathbb{R}$ be the map given by $f(z) = \Re(z)$, the real part of z, i.e. $f(a + ib) = a$ (where $a, b \in \mathbb{R}$). Prove that f is a linear map of $\mathbb{R}$-vector spaces. Is it a ring homomorphism? Justify your answers. ***(4 marks)***

(ii) Consider the First Isomorphism Theorem for f as a linear map of $\mathbb{R}$-vector spaces. What does it tell us in this instance? ***(3 marks)***

(iii) Let $g : \mathbb{C} \to \mathbb{C}$ be given by $g(z) = \overline{z}$, the complex conjugate. Prove that g is a linear map of $\mathbb{R}$-vector spaces. Is it a linear map of $\mathbb{C}$-vector spaces? Is it a ring homomorphism? Justify your answers. Write down the matrix representing the $\mathbb{R}$-linear operator g with respect to the basis $\{1, i\}$ for the $\mathbb{R}$-vector space $\mathbb{C}$. ***(6 marks)***

7 Consider the linear map $\theta : C([0, 2], \mathbb{R}) \to \mathbb{R}$ given by $\theta(f) := \int_0^2 f(x)\, dx$, where $C([0, 2], \mathbb{R})$ is the space of continuous real-valued functions on the interval $[0, 2]$. Find a non-zero element of the kernel $\ker(\theta)$. ***(3 marks)***

8 You may assume that $\langle f, g\rangle := \int_0^1 f(x)g(x)\, dx$ defines an inner product on the $\mathbb{R}$-vector space $C([0, 1], \mathbb{R})$ of continuous real-valued functions on the interval $[0, 1]$.

(i) Find $\langle 1, 1\rangle$, $\langle 1, x\rangle$ and $\langle x, x\rangle$. ***(1 mark)***

(ii) What are the length of x and the angle between 1 and x, with respect to this inner product? ***(2 marks)***

(iii) Without any further integration, find an element $f \in \mathrm{Span}\{1, x\}$ such that $\{1, f\}$ is an orthonormal basis for $\mathrm{Span}\{1, x\}$. ***(3 marks)***

End of Question Paper