MAS220 Exam 2014-15, [Annotated] Solutions

1 (i) The conjugacy class of x is $\{gxg^{-1} \mid g \in G\}$ [Several people did not use set notation correctly.]

(ii) G is abelian $\Leftrightarrow$ $gh = hg$ $\forall g, h \in G$.

(iii) If G is abelian, $gx = xg$, so $gxg^{-1} = x$ $\forall g \in G$, and the conjugacy class of x is just $\{x\}$. This is true $\forall x \in G$, so all conjugacy classes are of size 1. [See Sem. 1, Lect. 4]

2 (i) $\{(1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$

[Elements of S_n are conjugate in S_n if and only if they have the same "shape" when written as a product of disjoint cycles. See Sem. 1, Lect. 5. A common mistake was to include id. to make it look the same as V_4.]

(ii)

	id.	$(12)(34)$	$(13)(24)$	$(14)(23)$
id.	id.	$(12)(34)$	$(13)(24)$	$(14)(23)$
$(12)(34)$	$(12)(34)$	id.	$(14)(23)$	$(13)(24)$
$(13)(24)$	$(13)(24)$	$(14)(23)$	id.	$(12)(34)$
$(14)(23)$	$(14)(23)$	$(13)(24)$	$(12)(34)$	id.

[This was well done, with a few exceptions.]

We can check the subgroup criterion for V_4.

SG1: clearly $V_4 \neq \emptyset$.

SG2: the table shows that $g, h \in V_4 \Rightarrow gh \in V_4$.

SG3: it also shows that $g \in V_4 \Rightarrow g^{-1} \in V_4$, since in fact $g^{-1} = g$ (note the id.s down the diagonal).

Hence V_4 is a subgroup of S_4.

[Here it is important not just to state the subgroup criterion, but to give some explanation of how it is satisfied in this situation.]

It is normal because it is a union of the conjugacy classes $\{\text{id.}\}$ and $\{(1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$

[See Sem. 1, Lect. 5, Prop. 1. Again, avoid unjustified statements like "because the left and right cosets are the same". Although this shows that you are familiar with the definition in Sem. 1, Lect. 3, it gives no reason why V_4 actually satisfies this condition, and would not get any credit.]

$$|S_4/V_4| = \frac{|S_4|}{|V_4|} = \frac{4!}{4} = 6.$$

[This many cosets. Each of this size Make up this many elements]

[Anybody who subtracted is under a basic misapprehension of what the elements of a quotient group are. A quotient group is a collection of cosets.]

(iii) In $\mathbb{Z}/4\mathbb{Z}$, $\bar{1}$ has order 4, whereas in V_4 every non-identity element has order 2 (self-inverse). So $\mathbb{Z}/4\mathbb{Z}$ and V_4 cannot be isomorphic.
The subgroup $\langle(1\,2\,3\,4)\rangle$ of S_4 is cyclic of order 4, hence isomorphic to $\mathbb{Z}/4\mathbb{Z}$.

[Since I did not specify any operation on $\mathbb{Z}/4\mathbb{Z}$, you can assume it is the standard operation of addition:

$+$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{0}$	$\bar{1}$	$\bar{2}$

Quite a lot of people chose multiplication instead, but this is not an option, since $\bar{0}$ and $\bar{2}$ do not have multiplicative inverses, so $\mathbb{Z}/4\mathbb{Z}$ is not a group under multiplication. The subset $\{\bar{1}, \bar{3}\}$ of units is. See end of Sem. 1, Lect. 13.]

(iv) $a = (1\,2)$, $b = (1\,2\,3)$. $ab = (2\,3)$, $ba = (1\,3)$.

[That $ab \neq ba$ shows only that S_4 is non-abelian.
This is not enough in itself to show that the quotient S_4/V_4 is non-abelian.
For example, the quotient $S_4/A_4 \simeq \{\pm 1\}$ is abelian.
Sem. 1, Lect. 8, Example ②(i)
We need to show that $(aV_4)(bV_4) \neq (bV_4)(aV_4)$, i.e. that $abV_4 \neq baV_4$, i.e. that $(ab)(ba)^{-1} \notin V_4$.]

$(ab)(ba)^{-1} = (2\,3)(1\,3) = (1\,2\,3) \notin V_4$, so
$(aV_4)(bV_4) \neq (bV_4)(aV_4)$, so S_4/V_4 is non-abelian.
[Nobody got this!]

3 (i) $r.0 = r(0+0) = r.0 + r.0$. Adding $-(r.0)$ to both sides, $0 = r0$.
More carefully, $0 = (r0 + r0) + [-(r0)] = r0 + (r0 + (-(r0))) = r0 + 0 = r0$.

[Many assumed, without justification, that $(-r)0 = -(r0)$.]

(ii) $f : R \to S$ is a ring homomorphism $\Leftrightarrow$
① $f(a+b) = f(a)+f(b) \quad \forall a,b \in R$;
② $f(ab) = f(a)f(b) \quad \forall a,b \in R$;
③ $f(1_R) = 1_S$.

[It was a common mistake to omit the third condition, which is not a consequence of the second, since a ring is not a group under multiplication. In a ring it is important to distinguish clearly between the additive identity 0 and the multiplicative identity 1. Some people seemed to think they were doing groups, writing e_R and e_S, or even e_G and e_H.]

(iii) I is an ideal in R if
I is an additive subgroup of R
and $RI \subseteq I$ [Since R is commutative, $IR \subseteq I$ is superfluous.]
$\ker f = \{a \in R \mid f(a) = 0\}$ [Not $f(a)=1$!]
$f(0+0) = f(0)+f(0) \Rightarrow f(0)=0$ so $0 \in \ker f$.
$a, b \in \ker f \Rightarrow f(a+b) = f(a)+f(b) = 0+0 = 0$, so $a+b \in \ker f$.
$a \in \ker f \Rightarrow f(-a)+f(a) = f(-a+a) = f(0) = 0 \Rightarrow f(-a) = 0-0 = 0 \Rightarrow -a \in \ker f$.
Hence $\ker f$ is an additive subgroup of R [Naturally, since f is a homomorphism of additive groups, by condition ①.]
If $a \in \ker f$ and $r \in R$ then $f(ra) = f(r)f(a)$
$= f(r).0 = 0$ (by (i))
so $ra \in \ker f$. Hence $\ker f$ is an ideal.

Suppose that $\ker(f) = \{0\}$. If $f(a)=f(b)$ then $f(a-b) = f(a)-f(b) = 0$, so $a-b \in \ker f$, so $a-b=0$, so $a=b$. Hence f is injective.
[Common mistake to assume $f(a)=f(b)=0$.]
[A lot of people did not seem to like this kind of abstract reasoning, removed from specific examples. But without the abstraction we fail to see what unites disparate examples, and to take advantage of what they have in common.]

4 Congruence classes in R are represented by remainders $a+bx+cx^2$ with $a,b,c \in \mathbb{F}_2$, so it is a 3-dimensional $\mathbb{F}_2$-vector space with basis $\{\bar{1}, \bar{x}, \bar{x}^2\}$

x^3+x+1 is an irreducible element of the Euclidean domain $\mathbb{F}_2[x]$ (it has no roots in $\mathbb{F}_2$, so no linear factor), so R is a field, 1-dimensional as a vector space over itself.

[Nobody got full marks, and most got nothing, though it should have been OK for anybody who made sure they understood the beginning of Sem. 2, Lect. 5.]

5 (i) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$ has a multiplicative inverse in $M_2(\mathbb{R}) \iff ad-bc \neq 0$.

(ii) f_B is a bijection because it has an inverse function $f_{B^{-1}}$:

$$f_{B^{-1}}(f_B(A)) = B^{-1}(BAB^{-1})B = (B^{-1}B)A(B^{-1}B) = IAI = A,$$

and similarly $f_B(f_{B^{-1}}(A)) = A$.

If $B = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$, $f_B\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$

$$= \frac{1}{2}\begin{pmatrix} 5 & 8 \\ 8 & 12 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -6 & 11 \\ -8 & 16 \end{pmatrix} = \begin{pmatrix} -3 & 11/2 \\ -4 & 8 \end{pmatrix}.$$

[The inverse of an invertible 2x2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. There is no need to use row reduction to find it.]

(iii) We already know that f_B is a bijection, so to prove that it is an isomorphism, we just need to prove that it is a homomorphism, as follows.

$$f_B(A_1+A_2) = B(A_1+A_2)B^{-1} = (BA_1+BA_2)B^{-1} = BA_1B^{-1} + BA_2B^{-1} = f_B(A_1)+f_B(A_2)$$

$$f_B(A_1A_2) = BA_1A_2B^{-1} = BA_1B^{-1}BA_2B^{-1} = f_B(A_1)f_B(A_2).$$

$$f_B(I) = BIB^{-1} = BB^{-1} = I.$$

[Given the response to 3(ii), unsurprisingly this last bit was often omitted.]

(iv) $B^{-1} = \begin{pmatrix} 1 & -1/2 \\ -1 & 1 \end{pmatrix} \notin M_2(\mathbb{Z})$, so B is not invertible in $M_2(\mathbb{Z})$.

$C = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ is invertible in $M_2(\mathbb{Z})$, since $\det C = 1$ (the inverse is $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \in M_2(\mathbb{Z})$).

$D = \begin{bmatrix} 59 & b \\ 13 & d \end{bmatrix}$. Need $\det D = 1$, i.e. $59d - 13b = 1$.

Euclid's algorithm: $59 = 4\cdot 13 + 7$
$13 = 2\cdot 7 - 1$,

so $1 = 2\cdot 7 - 13 = 2(59 - 4\cdot 13) - 13 = 2\cdot 59 - 9\cdot 13$.

Take $d=2$, $b=9$, $D = \begin{bmatrix} 59 & 9 \\ 13 & 2 \end{bmatrix}$.

[It was OK to spot b, d by trial and error. We can get other solutions $b = 9 + 59n$, $d = 2 + 13n$, for any $n \in \mathbb{Z}$, and I saw a few of these in the scripts.]

6 (i) $f(z_1+z_2) = \mathrm{Re}(z_1+z_2) = \mathrm{Re}(z_1) + \mathrm{Re}(z_2) = f(z_1)+f(z_2) \quad \forall z_1, z_2 \in \mathbb{C}$

so f is a homomorphism of additive groups.

For $z \in \mathbb{C}$, $\lambda \in \mathbb{R}$, $f(\lambda z) = \mathrm{Re}(\lambda z) = \lambda \mathrm{Re}(z) = \lambda f(z)$, so f is a linear map of $\mathbb{R}$-vector spaces.

[I was pleased to see many people go into more detail than I did, letting $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$, with $a_1, a_2, b_1, b_2 \in \mathbb{R}$, then $f(z_1+z_2) = f((a_1+b_1 i)+(a_2+b_2 i))$
$= f((a_1+a_2)+(b_1+b_2)i) = a_1 + a_2 = f(z_1) + f(z_2)$.]

It is not a ring homomorphism, e.g. $f(i \cdot i) = f(-1) = -1$, whereas
$f(i)f(i) = 0.0 = 0$.

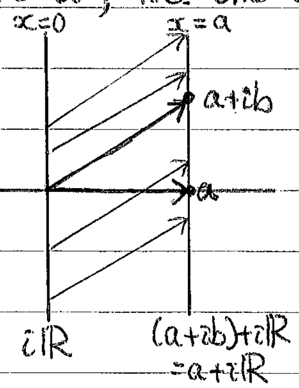
[One counterexample such as this is sufficient. But it was nice to see many people exploring what happens in general, with $f((a+bi)(c+di)) = ac - bd$, while $f(a+bi)f(c+di) = ac$, which is different whenever $b \neq 0$ and $d \neq 0$.]

(ii) F.I.T. for f as a linear map of $\mathbb{R}$-vector spaces says

$$\mathbb{C}/\ker f \underset{\substack{\uparrow \\ \text{isomorphism of} \\ \mathbb{R}\text{-vector spaces}}}{\cong} \mathrm{im}\, f, \quad \text{i.e. } \mathbb{C}/i\mathbb{R} \cong \mathbb{R}$$

$$[a+ib] \longmapsto a$$

[It is not enough to state the general F.I.T. You must see what it is saying in this particular situation, by identifying the kernel and image. This example provides a good illustration of what a quotient structure is. Elements of $\mathbb{C}/\ker f$ are not complex numbers. They are cosets of $\ker f$. $\ker f$ is the imaginary axis $i\mathbb{R}$, ~~The coset $(a+ib)+i\mathbb{R}$~~ i.e. the set of all complex numbers with real part 0. Letting $z = x+iy$, it has equation $x = 0$. The coset $[a+ib] = (a+ib)+i\mathbb{R} = a + i\mathbb{R}$ is the set of all complex numbers with real part a, i.e. the vertical line $x = a$.



We get from $x=0$ to $x=a$ by adding any complex number with real part a. They are all equivalent, i.e. in the same coset of $i\mathbb{R}$ (their difference is in $i\mathbb{R}$), i.e. on the same line $x = a$.

So $\mathbb{C}/\ker f$ is a collection of vertical lines, $\{L_a : a \in \mathbb{R}\}$, where L_a is the line $x = a$, i.e. the coset $[a+ib] = a + i\mathbb{R}$.

The operations of addition and scalar multiplication on $\mathbb{C}/\ker f$ are $[a+ib]+[c+id] = [(a+ib)+(c+id)] = [(a+c)+i(b+d)]$
and $\lambda[a+ib] = [\lambda(a+ib)] = [\lambda a + i\lambda b]$

i.e. $L_a + L_c = L_{a+c}$
and $\lambda L_a = L_{\lambda a}$

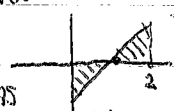
Clearly we can keep track of these lines by just looking at the subscripts — that is the isomorphism with $\mathbb{R}$.
Though elements of $\mathbb{C}/\ker f$ are not complex numbers, they are equivalence classes of complex numbers, where two complex numbers are considered equivalent if they have the same real part. The equivalence classes partitioning the complex plane $\mathbb{C}$ are the vertical lines L_a.
Though the details are of course different, this is exactly the same kind of thing as when we partitioned S_4 into cosets of V_4. The quotient group S_4/V_4 is likewise a collection of cosets. Maybe that is enough.]

(iii) $g(z_1+z_2) = \overline{z_1+z_2} = \overline{z_1}+\overline{z_2} = g(z_1)+g(z_2) \quad \forall z_1, z_2 \in \mathbb{C}$.
$g(\lambda z) = \overline{\lambda z} = \overline{\lambda}\,\overline{z} = \lambda \overline{z}$ (since $\lambda \in \mathbb{R}$ so $\overline{\lambda} = \lambda$)
$= \lambda g(z) \quad \forall z \in \mathbb{C}, \lambda \in \mathbb{R}$.
Hence g is a linear map of $\mathbb{R}$-vector spaces.
It is not $\mathbb{C}$-linear, since $g(\lambda z) = \overline{\lambda}\,\overline{z}$ while $\lambda g(z) = \lambda \overline{z}$, and if $\lambda \in \mathbb{R}$ these are not the same unless $z=0$.
E.g. $g(i.1) = \overline{i} = -i$, whereas $i\,g(1) = i\overline{1} = i$.

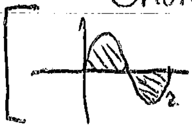
It is a ring homomorphism, since $g(z_1 z_2) = \overline{z_1 z_2} = \overline{z_1}\,\overline{z_2} = g(z_1)g(z_2)$
and $g(1) = \overline{1} = 1$.
[Again, it was nice to see people doing this in more detail with $z_1 = a_1+b_1 i$, $z_2 = a_2+b_2 i$, thus properly justifying assertions such as $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.]

The required matrix is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, [since $1 \mapsto 1 = 1.1 + 0.i$ and $i \mapsto -i = 0.1 + (-1)i$].

7 We are looking for non-zero $f \in C([0,2], \mathbb{R})$ [so nothing involving $1/x$, please] such that $\int_0^2 f(x)\,dx = 0$. Many possible answers, popular choices being $f(x) = x-1$



or $f(x) = \sin \pi x$. [



Area above and below x-axis cancels, as may also be confirmed by calculation.]

8 (i) $\langle 1, 1\rangle = \int_0^1 1.1\,dx = \int_0^1 1\,dx = [x]_0^1 = 1-0 = \underline{\underline{1}}$.

$\langle 1, x\rangle = \int_0^1 1.x\,dx = \int_0^1 x\,dx = \left[\frac{1}{2}x^2\right]_0^1 = \underline{\underline{1/2}}$

$\langle x, x\rangle = \int_0^1 x.x\,dx = \int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1 = \underline{\underline{1/3}}$

(ii) Length of $x = \sqrt{\langle x, x\rangle} = \underline{\underline{1/\sqrt{3}}}$

Angle between 1 and x is $\cos^{-1}\left(\dfrac{\langle 1, x\rangle}{\sqrt{\langle 1,1\rangle}\sqrt{\langle x,x\rangle}}\right) = \cos^{-1}\left(\dfrac{1/2}{1 \cdot 1/\sqrt{3}}\right)$

$= \cos^{-1}\left(\sqrt{3}/2\right) = \underline{\underline{\pi/6}}$

(iii) $f = a+bx$

Require $\langle 1, f\rangle = 0$ and $\langle f, f\rangle = 1$.

$\langle 1, f\rangle = 0 \Rightarrow \langle 1, a+bx\rangle = 0 \Rightarrow a\langle 1,1\rangle + b\langle 1, x\rangle = 0$

$\Rightarrow a + \frac{1}{2}b = 0 \Rightarrow a = -\frac{1}{2}b$, so $f = b(x-\frac{1}{2})$.

[Alternatively, Gram-Schmidt gives $x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle} 1 = x - \frac{1}{2}$.]

$\langle f, f\rangle = b^2\langle x-\frac{1}{2}, x-\frac{1}{2}\rangle = b^2(\langle x,x\rangle - \langle x,1\rangle + \frac{1}{4}\langle 1,1\rangle) = b^2(\frac{1}{3}-\frac{1}{2}+\frac{1}{4}) = b^2/12$.

So choose $b = \sqrt{12} = 2\sqrt{3}$, $f = \underline{\underline{\sqrt{3}(2x-1)}}$. [$b = -2\sqrt{3}$ works equally well]

[This is equivalent to taking $x - \frac{1}{2}$ and dividing by its length, $1/\sqrt{12}$, to get a vector of length 1.]