

annotated
MAS 220 Exam 2015-16 Solutions

1 (i) $\text{conj}_G(x) = \{g x g^{-1} \mid g \in G\}$

[Largely well-done, though some were unable to use set notation correctly.]

$\text{cent}_G(x) = \{g \in G \mid g x g^{-1} = x\}$
 ↑
 i.e. $g x = x g$

(ii) Applying β to the entries in α , $\beta \alpha \beta^{-1} = (54)(132)$.

$\text{Conj}_{S_5}(\alpha)$ contains $\binom{5}{2} \times 2 = \underline{\underline{20}}$ elements
 ↑ choices for 2 numbers to go in the transposition
 ↙ number of 3-cycles on the remaining 3 numbers

$(345) \in \text{cent}_{S_5}(\alpha)$. [So is anything in the subgroup of S_5 generated by (12) and (345) , which is $\text{cent}_{S_5}(\alpha)$]

$$|\text{cent}_{S_5}(\alpha)| = \frac{|S_5|}{|\text{conj}_{S_5}(\alpha)|} = \frac{5!}{20} = \frac{120}{20} = \underline{\underline{6}}$$

2 SG1 $e x = x e = x, \forall x \in G$, so $e \in Z(G)$

SG2 Suppose that $g, h \in Z(G)$, so $g x = x g$ and $h x = x h \quad \forall x \in G$.

Then $(gh)x = g(hx) = g(xh) = (gx)h = (xg)h = x(gh) \quad \forall x \in G$

So $gh \in Z(G)$.

SG3 Suppose that $g \in Z(G)$. For any $x \in G$, $g x = x g$, so $x = g^{-1} x g$, so $x g^{-1} = g^{-1} x$. Hence $g^{-1} \in Z(G)$.

We have shown that $Z(G)$ is a subgroup of G .

For any $x \in G$, $x Z(G) = \{x g \mid g \in Z(G)\} = \{g x \mid g \in Z(G)\} = Z(G) x$.

Hence $Z(G)$ is normal.

3 For any $n, m \in \mathbb{Z}$, $\theta(n+m) = J^{n+m} = J^n J^m = \theta(n)\theta(m)$

Hence θ is a group homomorphism.

[Many people applied the definition of homomorphism to decide that they were aiming at $\theta(nm) = \theta(n)\theta(m)$, failing to use "+" for the operation on the left to stop it looking like multiplication, then tried to tell me nonsense like $J^{nm} = J^n J^m$]

$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, J^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I, J^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, J^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$.

F.I.T. $\Rightarrow \mathbb{Z}/\ker \theta \cong \text{im } \theta$, i.e. $\mathbb{Z}/\langle 4\mathbb{Z} \rangle \cong \langle J \rangle = \{\pm I, \pm J\}$

$\frac{\mathbb{Z}}{n} \mapsto J^n$

[Some apparently ignored the last sentence in the question.]

$$\underline{4} \quad \theta((a+bi)+(c+di)) = \theta((a+c)+(b+d)i) = \begin{bmatrix} a+c & -(b+d) \\ b+d & a+c \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \\ = \theta(a+bi) + \theta(c+di)$$

$$\theta((a+bi)(c+di)) = \theta((ac-bd)+(ad+bc)i) = \begin{bmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{bmatrix}$$

$$\theta(a+bi)\theta(c+di) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{bmatrix}, \text{ the same.}$$

$$\theta(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I. \quad \text{Hence } \theta \text{ is a ring homomorphism.}$$

[Largely well done. Some forgot to check $\theta(1)$. Incidentally, notice that $\theta(i) = J$ from Q3.]

$\theta: \mathbb{C} \rightarrow M_2(\mathbb{R})$ is not a ring isomorphism because it is not surjective, e.g. $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \notin \text{im } \theta$. Alternatively, \mathbb{C} is commutative while $M_2(\mathbb{R})$ is not.

5 (i) R is a field \Leftrightarrow every non-zero element a has a multiplicative inverse b , such that $ab=1$.

(ii) \mathbb{F}_3 is a field because 3 is a prime number (an irreducible element of the Euclidean domain \mathbb{Z}). $5^{-1} = \bar{8}$.

(iii) $[x-5][x-8] = [(x-\bar{5})(x-\bar{8})] = [x^2 - \bar{13}x + \bar{40}] = [x^2 + \bar{1}] = [\bar{0}]$.

[Note that $\frac{\mathbb{F}_3[x]}{\langle x^2+1 \rangle}$, a quotient ring, has nothing to do with dividing by

x^2+1 . So the answer is not $[x^2+1] = \left\{ \frac{x^2+1}{x^2+1} \right\} = \bar{1}$. This comment is for those who think that $\frac{1}{3} \in \mathbb{Z}/3\mathbb{Z}$.]

R is not a field, since the above calculation shows that it is not an integral domain.

$\frac{\mathbb{F}_3[x]}{\langle x^2+1 \rangle}$ is a field, because x^2+1 is an irreducible element of the Euclidean domain $\mathbb{F}_3[x]$. If it were reducible, it would have a linear factor, hence a root in \mathbb{F}_3 , but $(\pm\bar{1})^2 + \bar{1} = \bar{2} \neq \bar{0}$, and $\bar{0}^2 + \bar{1} = \bar{1} \neq \bar{0}$, so there are no roots.

6 θ is not a ring homomorphism, since $\det(A+B) \neq \det A + \det B$ in general, e.g. with $A=B=I$, $4 \neq 1+1$.

[Many claimed that $\det(A+B)$ is in fact $\det(A) + \det(B)$. Others claimed erroneously that $\det(AB) \neq \det A \det B$. Some worked out a formula for $\det(A+B)$ and compared it with one for $\det A + \det B$. I accepted this.]

7 (i) S is linearly independent $\Leftrightarrow \sum_{i=1}^m \alpha_i v_i = 0$ only when $\alpha_1 = \dots = \alpha_m = 0$.

[Many seemed to have trouble grasping the meaning of a logical statement such as this, writing down something incorrect though superficially similar on the page.]

(ii) $\text{Span}(S) = \left\{ \sum_{i=1}^m \alpha_i v_i \mid \alpha_1, \dots, \alpha_m \in F \right\}$

(iii) $1 = \cos^2 x + \sin^2 x \in \text{Span}(S)$.

Suppose that $a \cos^2 x + b \sin^2 x + c \sin 2x = 0$ (must show $a=b=c=0$).

Putting $x=0$, $a=0$, so $b \sin^2 x + c \sin 2x = 0$.

Putting $x=\frac{\pi}{2}$, $b=0$, hence also $c=0$. So S is linearly independent.

[When there is a good hint, use it!]

$\{\cos^2 x, \sin^2 x, \cos 2x\}$ is linearly dependent, since

$$\cos 2x = \cos^2 x - \sin^2 x.$$

[Note that it is not enough to imitate the above and fail to show that $a=b=c=0$. That doesn't prove that they aren't. Also it is not enough to find a relation that works for some x . It must hold for all x .]

8 $\ker(\lambda) = \text{Null}(A) =$ solution space to homogeneous linear system $Ax=0$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

Solution space has basis $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$. [This kind of thing should be utterly routine.]

9 (i) $(BAB^{-1})Bv = BA(B^{-1}B)v = BA v = B \lambda v = \lambda (Bv)$.

(Also $v \neq 0$ and B invertible $\Rightarrow Bv \neq 0$). So Bv is an eigenvector for BAB^{-1} , with eigenvalue λ .

(ii) (a) F_A is a rotation about the x -axis, through an angle $\frac{\pi}{2}$.

F_B is a rotation about the z -axis, through an angle $\frac{\pi}{3}$.

(b) $\det(\lambda I - A) = (\lambda - 1)(\lambda^2 + 1) = (\lambda - 1)(\lambda - i)(\lambda + i)$

$\lambda = 1$ eigenspace is $\text{Null} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

Eigenvectors for $\lambda = \pm i$ are necessarily in $\mathbb{C}^3 - \mathbb{R}^3$.

Note that we already knew that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector with eigenvalue 1, since it is along the axis of rotation.

By (i), $B \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}$ is an eigenvector for BAB^{-1} .

[Some failed to apply "Using (i)", and instead calculated BAB^{-1} and found the eigenvector the long way. I accepted this.]

$f_{BAB^{-1}}$ is a rotation through $\frac{\pi}{2}$ about an axis along this vector.
[I don't recall anybody seeing this. I grudgingly accepted descriptions of doing three rotations, one after the other.]

10 $l(y) = 2y \Leftrightarrow x^2 \frac{dy}{dx} = 2y$ [Many failed to get this far, not grasping the meaning of y and $l(y)$ as elements of the vector space $C^\infty(\mathbb{R}_{>0}, \mathbb{R})$.]

$$\Leftrightarrow \int \frac{dy}{y} = \int \frac{2}{x^2} dx$$

$$\Leftrightarrow \ln|y| = -2/x + C \Leftrightarrow y = A e^{-2/x}$$
 [You are not allowed to forget MAS110!]

$e^{-2/x}$ is an eigenvector for l , with eigenvalue 2.

11 (i) It means that $\langle Tu, v \rangle = \langle u, Tv \rangle \quad \forall u, v \in V$.

(ii) $\lambda_1 \langle v_1, v_2 \rangle = \langle \lambda_1 v_1, v_2 \rangle = \langle Tv_1, v_2 \rangle = \langle v_1, Tv_2 \rangle = \langle v_1, \lambda_2 v_2 \rangle = \lambda_2 \langle v_1, v_2 \rangle$,
but $\lambda_1 \neq \lambda_2$, hence $\langle v_1, v_2 \rangle = 0$, i.e. v_1 and v_2 are orthogonal.

[Mostly well-remembered.]

(iii) $\langle Tx, y \rangle = (Ax)^t y = (x^t A^t) y = x^t (A^t y) = x^t (Ay)$ (since $A = A^t$)
 $= \langle x, Ay \rangle = \langle x, Ty \rangle$, $\forall x, y \in \mathbb{R}^n$, hence T is self-adjoint.

$$(iv) A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \det(xI - A) = \begin{vmatrix} x-2 & -1 \\ -1 & x-2 \end{vmatrix} = (x-2)^2 - 1 = x^2 - 4x + 3 = (x-1)(x-3).$$

Eigenvalues $\lambda=1, \lambda=3$.

$$\underline{\lambda=1} \quad A - \lambda I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{Null space Span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$$\underline{\lambda=3} \quad A - \lambda I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{Null space Span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

A is symmetric, so gives a self-adjoint operator.

The eigenvectors $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for the distinct eigenvalues 1, 3 are visibly orthogonal, in accord with (ii):
 $(1)(1) + (-1)(1) = 0$.

$$\underline{12} \text{ (i) } \langle 1, x \rangle = \int_{-1}^1 \frac{x}{\sqrt{1-x^2}} dx = \left[-(1-x^2)^{1/2} \right]_{-1}^1 = 0 - 0 = 0.$$

So $1, x$ are orthogonal

$$\|1\|^2 = \langle 1, 1 \rangle = \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} dx = \left[\sin^{-1} x \right]_{-1}^1 = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi, \text{ so } \|1\| = \underline{\underline{\sqrt{\pi}}}$$

$$\|x\|^2 = \langle x, x \rangle = \int_{-1}^1 \frac{x^2}{\sqrt{1-x^2}} dx \quad \left. \begin{array}{l} \text{Put } x = \sin \theta, \quad dx = \cos \theta d\theta \\ \sqrt{1-x^2} = \cos \theta \quad -\pi/2 \leq \theta \leq \pi/2 \end{array} \right\}$$

$$= \int_{-\pi/2}^{\pi/2} \frac{\sin^2 \theta}{\cancel{\cos \theta}} \cos \theta d\theta = \int_{-\pi/2}^{\pi/2} \frac{1 - \cos 2\theta}{2} d\theta = \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_{-\pi/2}^{\pi/2} = \frac{\pi}{2},$$

so $\|x\| = \underline{\underline{\sqrt{\pi/2}}}$.

$$\|3+2x\|^2 = 3^2 \|1\|^2 + 2^2 \|x\|^2 \quad (\text{since } \langle 1, x \rangle = 0)$$

$$= 9\pi + 4 \frac{\pi}{2} = 11\pi, \text{ so } \|3+2x\| = \underline{\underline{\sqrt{11\pi}}}.$$

[I was lenient about forgetting to square root.]

(A) Axiom IP4 is not satisfied. Many ways to see this, e.g.

$$\langle x, x \rangle = \int_{-1}^1 x^3 dx = \left[\frac{x^4}{4} \right]_{-1}^1 = 0, \text{ even though } x \neq 0.$$

[This means the function $x: [-1, 1] \rightarrow \mathbb{R}$ is not the same as the function $0: [-1, 1] \rightarrow \mathbb{R}$. The fact that they take the same value at $0 \in [-1, 1]$ is irrelevant.]