MAS 220 Exam 2015-16 annotated Solutions

1 (i) $\mathrm{conj}_G(x) = \{gxg^{-1} \mid g \in G\}$ [Largely well-done, though some were unable to use set notation correctly.]

$\mathrm{cent}_G(x) = \{g \in G \mid gxg^{-1} = x\}$

i.e. $gx = xg$

(ii) Applying β to the entries in α, $\beta\alpha\beta^{-1} = (5\,4)(1\,3\,2)$.

$\mathrm{Conj}_{S_5}(\alpha)$ contains $\binom{5}{2} \times 2 = \underline{\underline{20}}$ elements

choices for 2 numbers to go in the transposition

number of 3-cycles on the remaining 3 numbers

$(3\,4\,5) \in \mathrm{cent}_{S_5}(\alpha)$. [So is anything in the subgroup of S_5 generated by $(1\,2)$ and $(3\,4\,5)$, which is $\mathrm{cent}_{S_5}(\alpha)$.]

$$|\mathrm{cent}_{S_5}(\alpha)| = \frac{|S_5|}{|\mathrm{conj}_{S_5}(\alpha)|} = \frac{5!}{20} = \frac{120}{20} = \underline{\underline{6}}$$

2 <u>SG1</u> $ex = xe = x$, $\forall x \in G$, so $e \in Z(G)$

<u>SG2</u> Suppose that $g, h \in Z(G)$, so $gx = xg$ and $hx = xh$ $\forall x \in G$.

Then $(gh)x = g(hx) = g(xh) = (gx)h = (xg)h = x(gh)$ $\forall x \in G$

So $gh \in Z(G)$.

<u>SG3</u> Suppose that $g \in Z(G)$. For any $x \in G$, $gx = xg$, so $x = g^{-1}xg$, so $xg^{-1} = g^{-1}x$. Hence $g^{-1} \in Z(G)$.

We have shown that $Z(G)$ is a subgroup of G.

For any $x \in G$, $xZ(G) = \{xg \mid g \in Z(G)\} = \{gx \mid g \in Z(G)\} = Z(G)x$.

Hence $Z(G)$ is normal.

3 For any $n, m \in \mathbb{Z}$, $\theta(n+m) = J^{n+m} = J^n J^m = \theta(n)\theta(m)$

Hence θ is a group homomorphism.

[Many people applied the definition of homomorphism to decide that they were aiming at $\theta(nm) = \theta(n)\theta(m)$, failing to use "+" for the operation on the left to stop it looking like multiplication, then tried to tell me nonsense like $J^{nm} = J^n J^m$]

$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $J^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$, $J^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -J$, $J^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$.

F.I.T. $\Rightarrow \mathbb{Z}/_{\ker\theta} \cong \mathrm{im}\,\theta$, i.e. $\mathbb{Z}/_{4\mathbb{Z}} \cong \langle J \rangle = \{\pm I, \pm J\}$

$\bar{n} \mapsto J^n$

[Some apparently ignored the last sentence in the question.]

4 $\theta((a+bi)+(c+di)) = \theta((a+c)+(b+d)i) = \begin{bmatrix} a+c & -(b+d) \\ b+d & a+c \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$

$= \theta(a+bi) + \theta(c+di)$

$\theta((a+bi)(c+di)) = \theta((ac-bd)+(ad+bc)i) = \begin{bmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{bmatrix}$

$\theta(a+bi)\,\theta(c+di) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{bmatrix}$, the same.

$\theta(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$. Hence θ is a ring homomorphism.

[Largely well done. Some forgot to check $\theta(1)$. Incidentally, notice that $\theta(i) = J$ from Q3.]

$\theta : \mathbb{C} \to M_2(\mathbb{R})$ is not a ring isomorphism because it is not surjective, e.g. $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \notin \operatorname{im}\theta$. Alternatively, $\mathbb{C}$ is commutative while $M_2(\mathbb{R})$ is not.

5 (i) R is a field $\Leftrightarrow$ every non-zero element a has a multiplicative inverse b, such that $ab = 1$.

(ii) $\mathbb{F}_{13}$ is a field because 13 is a prime number (an irreducible element of the Euclidean domain $\mathbb{Z}$). $\bar{5}^{-1} = \bar{8}$.

(iii) $[x-\bar{5}][x-\bar{8}] = [(x-\bar{5})(x-\bar{8})] = [x^2 - \bar{13}x + \bar{40}] = [x^2+\bar{1}] = [\bar{0}]$.

[Note that $\frac{\mathbb{F}_{13}[x]}{\langle x^2+1 \rangle}$, a quotient ring, has nothing to do with dividing by $x^2+\bar{1}$. So the answer is not $[x^2+\bar{1}] = \left\{\frac{x^2+\bar{1}}{x^2+\bar{1}}\right\} = \bar{1}$. This comment is for those who think that $\frac{1}{3} \in \mathbb{Z}/3\mathbb{Z}$.]

R is not a field, since the above calculation shows that it is not an integral domain.

$\frac{\mathbb{F}_3[x]}{\langle x^2+\bar{1} \rangle}$ is a field, because $x^2+\bar{1}$ is an irreducible element of the Euclidean domain $\mathbb{F}_3[x]$. If it were reducible, it would have a linear factor, hence a root in $\mathbb{F}_3$, but $(\pm\bar{1})^2+\bar{1} = \bar{2} \neq \bar{0}$, and $\bar{0}^2+\bar{1} = \bar{1} \neq \bar{0}$, so there are no roots.

6 θ is not a ring homomorphism, since $\det(A+B) \neq \det A + \det B$ in general, e.g. with $A = B = I$, $4 \neq 1+1$.

[Many claimed that $\det(A+B)$ is in fact $\det(A)+\det(B)$. Others claimed erroneously that $\det(AB) \neq \det A \det B$. Some worked out a formula for $\det(A+B)$ and compared it with one for $\det A + \det B$. I accepted this.]

7 (i) S is linearly independent $\iff \sum_{i=1}^{m} \alpha_i v_i = 0$ only when $\alpha_1 = \dots = \alpha_m = 0$.

[Many seemed to have trouble grasping the meaning of a logical statement such as this, writing down something incorrect though superficially similar on the page.]

(ii) $\mathrm{Span}(S) = \{ \sum_{i=1}^{m} \alpha_i v_i \mid \alpha_1, \dots, \alpha_m \in F \}$

(iii) $1 = \cos^2 x + \sin^2 x \in \mathrm{Span}(S)$.

Suppose that $a\cos^2 x + b\sin^2 x + c\sin 2x = 0$ (must show $a=b=c=0$).

Putting $x=0$, $a=0$, so $b\sin^2 x + c\sin 2x = 0$.

Putting $x = \pi/2$, $b=0$, hence also $c=0$. So S is linearly independent.

[When there is a good hint, use it!]

$\{\cos^2 x, \sin^2 x, \cos 2x\}$ is linearly dependent, since

$$\cos 2x = \cos^2 x - \sin^2 x.$$

[Note that it is not enough to imitate the above and fail to show that $a=b=c=0$. That doesn't prove that they aren't. Also it is not enough to find a relation that works for some x. It must hold for all x.]

8 $\ker(\ell) = \mathrm{Null}(A) =$ solution space to homogeneous linear system $A\underline{x} = \underline{0}$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

Solution space has basis $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$. [This kind of thing should be utterly routine.]

9 (i) $(BAB^{-1})Bv = BA(B^{-1}B)v = BAv = B\lambda v = \lambda(Bv)$.

(Also $v \neq 0$ and B invertible $\Rightarrow Bv \neq \underline{0}$). So Bv is an eigenvector for BAB^{-1}, with eigenvalue λ.

(ii) (a) f_A is a rotation about the x-axis, through an angle $\pi/2$.

f_B is a rotation about the z-axis, through an angle $\pi/3$.

(b) $\det(\lambda I - A) = (\lambda-1)(\lambda^2+1) = (\lambda-1)(\lambda-i)(\lambda+i)$

$\lambda = 1$ eigenspace is $\mathrm{Null}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

Eigenvectors for $\lambda = \pm i$ are necessarily in $\mathbb{C}^3 - \mathbb{R}^3$.

Note that we already knew that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector with eigenvalue 1, since it is along the axis of rotation.

By (i), $B\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1/2\\ \sqrt{3}/2\\ 0\end{bmatrix}$ is an eigenvector for BAB^{-1}.

[Some failed to apply "Using (i)", and instead calculated BAB^{-1} and found the eigenvector the long way. I accepted this.]

$f_{BAB^{-1}}$ is a rotation through $\pi/2$ about an axis along this vector.

[I don't recall anybody seeing this. I grudgingly accepted descriptions of doing three rotations, one after the other.]

10 $\ell(y) = 2y \Leftrightarrow x^2 \dfrac{dy}{dx} = 2y$ [Many failed to get this far, not grasping the meaning of y and $\ell(y)$ as elements of the vector space $C^\infty(\mathbb{R}_{>0}, \mathbb{R})$.]

$\Leftrightarrow \displaystyle\int \frac{dy}{y} = \int \frac{2}{x^2}\,dx$

$\Leftrightarrow \ln|y| = -2/x + C \Leftrightarrow y = Ae^{-2/x}$ [You are not allowed to forget MAS110 !]

$e^{-2/x}$ is an eigenvector for ℓ, with eigenvalue 2.

11 (i) It means that $\langle Tu, v\rangle = \langle u, Tv\rangle \quad \forall u, v \in V$.

(ii) $\lambda_1\langle v_1, v_2\rangle = \langle \lambda_1 v_1, v_2\rangle = \langle Tv_1, v_2\rangle = \langle v_1, Tv_2\rangle = \langle v_1, \lambda_2 v_2\rangle = \lambda_2\langle v_1, v_2\rangle$,

but $\lambda_1 \neq \lambda_2$, hence $\langle v_1, v_2\rangle = 0$, i.e. v_1 and v_2 are orthogonal.

[Mostly well-remembered.]

(iii) $\langle T\underline{x}, \underline{y}\rangle = (A\underline{x})^t\,\underline{y} = (\underline{x}^t A^t)\underline{y} = \underline{x}^t(A^t\underline{y}) = \underline{x}^t(A\underline{y})$ (since $A = A^t$)

$= \langle \underline{x}, A\underline{y}\rangle = \langle \underline{x}, T\underline{y}\rangle$, $\forall \underline{x}, \underline{y} \in \mathbb{R}^n$, hence T is self-adjoint.

(iv) $A = \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$. $\det(xI - A) = \begin{vmatrix}x-2 & -1\\ -1 & x-2\end{vmatrix} = (x-2)^2 - 1 = x^2 - 4x + 3$
$= (x-1)(x-3)$.

Eigenvalues $\lambda = 1$, $\lambda = 3$.

$\underline{\lambda=1}$ $A - \lambda I = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ Null space $\mathrm{Span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$

$\underline{\lambda=3}$ $A - \lambda I = \begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}$ Null space $\mathrm{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$

A is symmetric, so gives a self-adjoint operator.

The eigenvectors $\begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$ for the distinct eigenvalues 1, 3 are visibly orthogonal, in accord with (ii):
$(1)(1) + (-1)(1) = 0$.

12. (i) $\langle 1, x\rangle = \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\,dx = \left[-(1-x^2)^{1/2}\right]_{-1}^{1} = 0-0 = 0.$

So $1, x$ are orthogonal

$\|1\|^2 = \langle 1, 1\rangle = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = \left[\sin^{-1} x\right]_{-1}^{1} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$, so $\|1\| = \underline{\underline{\sqrt{\pi}}}$

$\|x\|^2 = \langle x, x\rangle = \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}}\,dx$ Put $x = \sin\theta$, $dx = \cos\theta\, d\theta$, $\sqrt{1-x^2} = \cos\theta$, $-\pi/2 \le \theta \le \pi/2$

$= \int_{-\pi/2}^{\pi/2} \frac{\sin^2\theta}{\cancel{\cos\theta}} \cancel{\cos\theta}\, d\theta = \int_{-\pi/2}^{\pi/2} \frac{1-\cos 2x}{2}\,dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_{-\pi/2}^{\pi/2} = \pi/2,$

so $\|x\| = \underline{\underline{\sqrt{\pi/2}}}$.

$\|3+2x\|^2 = 3^2\|1\|^2 + 2^2\|x\|^2$ (since $\langle 1, x\rangle = 0$)

$= 9\pi + 4\,\pi/2 = 11\pi$, so $\|3+2x\| = \underline{\underline{\sqrt{11\pi}}}$.

[I was lenient about forgetting to square root.]

(ii) Axiom IP4 is not satisfied. Many ways to see this, e.g.

$\langle x, x\rangle = \int_{-1}^{1} x^3\,dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = 0$, even though $x \neq 0$.

[This means the function $x : [-1, 1] \to \mathbb{R}$ is not the same as the function $0 : [-1, 1] \to \mathbb{R}$. The fact that they take the same value at $0 \in [-1, 1]$ is irrelevant.]