MAS220, Semester 2 Problems.

1. (a) Let $A, B \in M_n(\mathbb{R})$ be invertible. Prove that AB is invertible, with $(AB)^{-1} = B^{-1}A^{-1}$.

   (b) Let $A \in M_{m,n}(\mathbb{R}), B \in M_{n,p}(\mathbb{R})$. By considering the ij entry on each side, show that $(AB)^T = B^T A^T$.

   (c) Let $O_n := \{A \in M_n(\mathbb{R}) \,:\, A^T A = I\}$. Prove that O_n is a group under matrix multiplication. (It is called the orthogonal group.) [Hint: $A^T A = I$ shows that A is invertible, with $A^{-1} = A^T$, which you must show also belongs to O_n.]

   (d) What is O_1? Give examples of elements $A, B \in O_2$ such that $AB \neq BA$.

   (e) Prove that if $A \in O_n$ then $\det(A) = \pm 1$. Give examples of elements $A, B \in O_2$ with $\det(A) = 1$ and $\det(B) = -1$.

   (f) For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, let $\mathbf{x}\cdot\mathbf{y} := \mathbf{x}^T\mathbf{y}$ (ignoring the brackets on the 1-by-1 matrix, so $\mathbf{x}\cdot\mathbf{y} \in \mathbb{R}$). Prove that if $A \in O(n)$ then $(A\mathbf{x})\cdot(A\mathbf{y}) = \mathbf{x}\cdot\mathbf{y}$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

   (g) Conversely, show that if $(A\mathbf{x}) \cdot (A\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then $A \in O(n)$. You might like to consider choosing $\mathbf{x} = e_i$, $\mathbf{y} = e_j$, where e_i is the vector with 1 in the i^{th} position and 0 elsewhere.

   (h) What has O_2 got to do with the group we called O_2 in Semester 1 (or in MAS114)?

2. (a) We define matrices $\mathfrak{i}, \mathfrak{j}, \mathfrak{k} \in M_2(\mathbb{C})$ by

$$\mathfrak{i} := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; \mathfrak{j} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; \mathfrak{k} := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

   Calculate $\mathfrak{i}^2, \mathfrak{j}^2, \mathfrak{k}^2, \mathfrak{i}\mathfrak{j}, \mathfrak{j}\mathfrak{i}, \mathfrak{j}\mathfrak{k}, \mathfrak{k}\mathfrak{j}, \mathfrak{k}\mathfrak{i}$ and $\mathfrak{i}\mathfrak{k}$.

   (b) Define a ring homomorphism $\theta : \mathbb{H} \to M_2(\mathbb{C})$, where $\mathbb{H} = \{a + bi + cj + dk \,:\, a, b, c, d \in \mathbb{R}\}$ is Hamilton's quaternion ring. (You must say what θ is, but you need not check in detail that it is a ring homomorphism.) Is θ injective? Is it surjective?

   (c) Given $\alpha = a + bi + cj + dk \in \mathbb{H}$, what is $\det(\theta(\alpha))$?

3. Let $P = \operatorname{Span}\left\{ \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\} = \left\{ \lambda \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \,:\, \lambda, \mu \in \mathbb{R} \right\}$, the unique plane through the origin containing the points $(5, -1, -1)$ and $(1, 1, -1)$. Express P in the form $ax + by + cz = 0$. [Hint: find a set of linear equations for a, b, c, and use row reduction to solve them.]

4. (deleted)

5. Let V be an F-vector space, and let U, W be subspaces of V. We define the *sum*

$$U + W := \{u + w \,:\, u \in U, w \in W\}.$$

   (a) Prove that $U + W$ is a subspace of V.

   (b) Let $V = \mathbb{R}^4$, with subspaces $U = \operatorname{Span}\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right\}$ and $W = \operatorname{Span}\left\{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\}$. What are $\dim(U)$ and $\dim(W)$? (The dimension of a vector space is the number of elements in a basis.) Show that $\begin{pmatrix}1\\2\\3\\0\end{pmatrix} \in U+W$, and find two different expressions $\begin{pmatrix}1\\2\\3\\0\end{pmatrix} = u+w$, with $u \in U, w \in W$. Find some $v \in V$ with $v \notin U + W$. What is $\dim(U + W)$ in this example?

   (c) Now find 2-dimensional subspaces U and W of $\mathbb{R}^4$ such that $U+W = \mathbb{R}^4$.

6. Let V be an F-vector space, and let U, W be subspaces of V.

   (a) Prove that $U \cap W$ is a subspace of V.

   (b) Let $V = \mathbb{R}^6$. Find 3-dimensional subspaces U_i, W_i for $1 \leq i \leq 4$ such that

$$\dim(U_1+W_1) = 3, \dim(U_2+W_2) = 4, \dim(U_3+W_3) = 5, \dim(U_4+W_4) = 6.$$

   [Hint: keep it simple. Try spans of subsets of the standard basis for $\mathbb{R}^6$.]

   (c) In each of your examples, what is $\dim(U_i \cap W_i)$?

   (d) In general, can you guess a relationship between the dimensions of the subspaces U, W, $U + W$ and $U \cap W$ (assuming they are finite-dimensional)?

7. (a) Let $v_1 = \begin{pmatrix}1\\1\\2\end{pmatrix}$, $v_2 = \begin{pmatrix}2\\1\\1\end{pmatrix}$, $v_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}$ and $v_4 = \begin{pmatrix}1\\2\\3\end{pmatrix}$. Why must this subset of $\mathbb{R}^3$ be linearly dependent? Find all the possible linear dependence relations $x_1v_1+x_2v_2+x_3v_3+x_4v_4 = 0$. [Hint: the possible $\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$ form the null space of a certain matrix A, which you should

row reduce to solve for x_1, x_2, x_3 in terms of x_4.] Hence express each v_i as a linear combination of the others. (Don't forget this bit!)

(b) Now, let $v_1 = \begin{pmatrix}1\\1\\2\end{pmatrix}$, $v_2 = \begin{pmatrix}2\\1\\1\end{pmatrix}$, $v_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}$ and $v_4 = \begin{pmatrix}2\\2\\3\end{pmatrix}$. Again, find all the possible linear dependence relations $x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 = 0$. Is it still the case that each v_i can be expressed as a linear combination of the others?

8. Let $L = \mathrm{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}\right\}$. Express this line through the origin in $\mathbb{R}^3$ also as Null(A) for some matrix $A \in M_{2,3}(\mathbb{R})$. [Hint: look at just one row $[a, b, c]$ of A, and find all the possibilities for $[a, b, c]$ in order to produce more than one row.] Now find a second, completely different, pair of homogeneous linear equations describing the same line.

9. Let C be the subspace Null(H) of $\mathbb{F}_2^7$, where $H = \begin{pmatrix}0&0&0&1&1&1&1\\0&1&1&0&0&1&1\\1&0&1&0&1&0&1\end{pmatrix}$.

(a) Find a basis for C. (You may write the vectors as horizontal strings.) What is its dimension? Could you have predicted this in advance?

(b) Putting these basis strings in the rows of a matrix G, and considering $\mathbf{x}G$ for all possible $\mathbf{x} \in (\mathbb{F}_2)_4$, list all the elements of C. (You could construct a table of $\mathbf{x}$ and $\mathbf{x}G$.) What is the smallest number of 1s in a non-zero element of C? Call this number d.

(c) How many pairs $c_1, c_2 \in C$ with $c_1 \neq c_2$ are there? Without checking them all, prove that such c_1 and c_2 must differ in at least d places. [Hint: consider their difference.]

(d) If $v = 1000101$, show that $v \notin C$. Find an element c of C that differs from v in precisely one place. Why must c be unique with this property? How does Hv tell you at which place c and v differ?

(e) If a string of 0s and 1s is broken up into blocks $\mathbf{x}$ of length 4, each $\mathbf{x}$ can be replaced by an element $c = \mathbf{x}G$ of C, now of length 7. Now if an error occurs in c, at one place, producing some v, we can correct the error by finding the unique element of C differing from v in only one place, thus turning v back into c. Then $\mathbf{x}$ can be recovered from c. For the c in the previous part, find $\mathbf{x}$.

10. Let V,W be finite-dimensional vector spaces over a field F, with $\dim(V) = n$, $\dim(W) = m$. We define the *direct sum*

$$V \oplus W := \{(v, w) \,:\, v \in V,\, w \in W\}.$$

Note that as a set this is the same as the cartesian product $V \times W$, seen in MAS110. We define operations of addition and scalar multiplication on

$V \oplus W$ by

$$(v_1, w_1) + (v_2, w_2) := (v_1 + v_2, w_1 + w_2) \ \ \forall v_1, v_2 \in V,\ w_1, w_2 \in W;$$

$$\lambda(v, w) := (\lambda v, \lambda w) \ \ \forall \lambda \in F,\ v \in V, w \in W.$$

(a) Prove carefully that with these operations, $V \oplus W$ is a vector space.

(b) Prove that if $\{v_1, v_2, \ldots, v_n\}$ is a basis for V and $\{w_1, \ldots, w_m\}$ is a basis for W, then $\{(v_1, 0), \ldots, (v_n, 0), (0, w_1), \ldots, (0, w_m)\}$ is a basis for $V \oplus W$. What then is $\dim(V \oplus W)$?

11. Let V be a vector space over a field F, and let U, W be subspaces of V.

(a) Show that the map $f : U \oplus W \to U + W$ defined by $f((u, w)) := u + w$ is linear.

(b) Suppose that $f((u_1, w_1)) = f((u_2, w_2))$. Prove that $u_1 - u_2 \in U \cap W$.

(c) Deduce that if $U \cap W = \{0\}$ then f is an isomorphism of vector spaces and $\dim(U + W) = \dim(U) + \dim(W)$ (if U and W are finite-dimensional).

(d) Prove that if $\dim(U) + \dim(W) > \dim(V)$ then $U \cap W \neq \{0\}$.

(e) In $\mathbb{R}^3$, let $U = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ and $W = \operatorname{Span}\left\{ \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$. Why must $\dim(U \cap W) = 1$? Find $v \in \mathbb{R}^3$ such that $U \cap W = \operatorname{Span}\{v\}$.

12. Let P be the plane $x + 3y + 4z = 0$ in $\mathbb{R}^3$.

(a) Of what matrix is P the null-space, i.e. P is the solution space of a system of homogeneous linear equations with what coefficient matrix?

(b) Find vectors $v_1, v_2 \in \mathbb{R}^3$ such that $P = \operatorname{Span}\{v_1, v_2\}$.

(c) If $f : P \to \mathbb{R}$ is a linear function (i.e. a linear map between these $\mathbb{R}$-vector spaces), prove that for some $a, b \in \mathbb{R}$, $f(sv_1 + tv_2) = as + bt$.

(d) Using $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = sv_1 + tv_2$, with the v_1, v_2 you found in (b), substitute for x, y and z in terms of s and t, and hence show that the linear functions $x - 3y + 2z$ and $3x + 3y + 10z$ are the same when restricted to P. How else could we have seen this? [Hint: consider the difference of these functions.]

(e) Evaluate the functions $x - 3y + 2z$ and $3x + 3y + 10z$ at the point $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. Now express $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ in the form $sv_1 + tv_2$, hence evaluate the same functions in a different way at this point, using what you found in (d).

(f) Find all possible $[a, b, c] \in \mathbb{R}_3$ such that the restriction to P of the linear function $ax + by + cz$ on $\mathbb{R}^3$ is the linear function $2s + 3t$ on P. Would the same method work for any linear function of the form $ds + et$, e.g. $-s + 2t$?

13. Let $f : V \to W$ be a linear map. Prove carefully that if f is a bijection then the inverse map $f^{-1} : W \to V$ is also linear. [Hint: given $w \in W$, write it as $f(v)$ for some $v \in V$.]

14. Let V be a vector space over a field F. Given any fixed $\mu \in F$, let $\theta_\mu : V \to V$ be the map $v \mapsto \mu v$. Prove that θ_μ is a linear map, justifying each step carefully. Prove also that if $\dim(V) = 1$ then every linear map $f : V \to V$ is necessarily of the form θ_μ, for some μ. [Hint: let $\{v\}$ be a basis for V. Now $f(v) \in V$, but $\{v\}$ is a basis for V, so $f(v) = \mu v = \theta_\mu(v)$ for some $\mu \in F$. You must show that with the same μ, $f(w) = \mu w$ for *any* $w \in V$, not just for $w = v$.]

15. While $\mathbb{R}^2$ is 2-dimensional as an $\mathbb{R}$-vector space (i.e. $\dim_{\mathbb{R}}(\mathbb{R}^2) = 2$), via the usual bijection $\mathbb{R}^2 \simeq \mathbb{C}$ ($\begin{pmatrix} a \\ b \end{pmatrix} \mapsto a + ib$) we may view it as the 1-dimensional $\mathbb{C}$-vector space $\mathbb{C}$ ($\dim_{\mathbb{C}}(\mathbb{R}^2) = \dim_{\mathbb{C}}(\mathbb{C}) = 1$).

    (a) Prove that the $\mathbb{R}$-linear map $\mathrm{rot}_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$, represented by the matrix $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$, is also $\mathbb{C}$-linear as a map from $\mathbb{C}$ to $\mathbb{C}$. [Hint: Referring to the previous question, what is μ, i.e. what complex number do you multiply by to effect this rotation? You might like to recall the polar form of a complex number.]

    (b) Find two linearly independent eigenvectors for the matrix $\begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix}$ representing the $\mathbb{R}$-linear map $\mathrm{ref}_\beta : \mathbb{R}^2 \to \mathbb{R}^2$. [Hint: $\beta = 2(\beta/2)$.] Is ref_β also $\mathbb{C}$-linear? In terms of complex numbers, how else might you describe ref_0?

    (c) What are all the possible bases for the 1-dimensional $\mathbb{C}$-vector space $\mathbb{C}$?

16. (deleted)

17. (a) If F is any field and X is a finite non-empty set, what is the dimension of the F-vector space $\mathcal{F}(X, F)$ of all functions from X to F? [Hint; what happened when $X = \{1, 2, \ldots, n\}$?]

    (b) More generally, if X is a finite non-empty set and V is a finite-dimensional vector space over F, with $\dim(V) = m$, what is $\dim(\mathcal{F}(X, V))$?

18. Let V, W be vector spaces over a field F, and let $V^* = L(V, F)$ and $W^* = L(W, F)$ be the dual spaces (of linear functions on V and on W). Let $\theta : V \to W$ be a linear map. We define a map $\theta^* : W^* \to V^*$ by

$$\theta^*(g) := g \circ \theta, \quad \forall g \in W^*,$$

i.e. $(\theta^*(g))(v) := g(\theta(v))$, $\forall g \in W^*, v \in V$. (You might like to draw a diagram to see what is going on.) Since a composition of linear maps is linear, we know that $\theta^*(g)$ really does belong to V^*, i.e. that $g \circ \theta$ is a *linear* map from V to F.

(a) Prove that $\theta^* : W^* \to V^*$ is linear, i.e. that $\theta^*(g_1 + g_2) = \theta^*(g_1) + \theta^*(g_2)$ and $\theta^*(\lambda g) = \lambda\theta^*(g)$ ($\forall$ etc.).

(b) In the case that $V = F^n$ and $W = F^m$, with θ given by the matrix $A \in M_{m,n}(F)$ (i.e. $\theta = \ell_A$), show that if $g \in W^*$ is the linear function specified by $\mathbf{a} \in F_m$ (i.e. $g = \ell_{\mathbf{a}}$, so $g(\mathbf{x}) = \mathbf{a}\mathbf{x} = \sum_{j=1}^m a_j x_j$), then $\theta^*(g)$ is associated with $\mathbf{a}A \in F_n$ (i.e. $\theta^*(g) = \ell_{\mathbf{a}A}$).

(c) Note that if V is a subspace of W and θ is the inclusion of V in W, then θ^* is just the restriction of linear functions from W to the subspace V. We had an example of this in Question 12, with $W = \mathbb{R}^3$ and $V = P$, the plane $x + 3y + 4z = 0$. Since $\begin{pmatrix} s \\ t \end{pmatrix} \mapsto sv_1 + tv_2$ is an isomorphism of $\mathbb{R}^2$ with P, we really have $\theta : \mathbb{R}^2 \to \mathbb{R}^3$, $\theta\left(\begin{pmatrix} s \\ t \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Using the v_1, v_2 you found in Question 12, give the matrix A such that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\begin{pmatrix} s \\ t \end{pmatrix}$. In Question 12(d) you calculated the restriction to P of the linear functions $x - 3y + 2z$ and $3x + 3y + 10z$. Recover the same answers using (b).

19. Let $D := x\frac{d}{dx}$.

(a) First look at D as a linear operator on $\mathbb{R}[x]$. What are its eigenvalues and eigenvectors? (If $\theta \in L(V)$, and if $\theta(v) = \lambda v$, with $0 \neq v \in V$, then by definition v is an eigenvector for θ, with eigenvalue λ.)

(b) Calculate an expression for D^2 in terms of x and $\frac{d}{dx}$, and check directly that it does what you would expect to the the functions you found in (a).

(c) Now look at D as a linear operator on $C^\infty(\mathbb{R}, \mathbb{R})$. Show that there is no eigenvector with eigenvalue $\lambda = -1$ or $\lambda = 1/2$. What if we restrict to $C^\infty([1, 2], \mathbb{R})$?

(d) Find, in $C^1(\mathbb{R}, \mathbb{R})$, an eigenvector for D, with eigenvalue $\lambda = 3$, that you did not already have in the subspace $\mathbb{R}[x]$.

20. (a) Prove that the operator $D = x\frac{d}{dx}$ on $C^\infty(\mathbb{R}, \mathbb{R})$ (i.e. $D(y) = x\frac{dy}{dx}$) is linear.

(b) What goes wrong if you try to prove that the operator $D' : y \mapsto y\frac{dy}{dx}$ on $C^\infty(\mathbb{R}, \mathbb{R})$ is linear?

21. Let $V = C^\infty(\mathbb{R}, \mathbb{R})$, and let $U = \{f \in V : f(x + 2\pi) = f(x)\ \forall\, x \in \mathbb{R}\}$.

   (a) Show that U is a subspace of V. Give an example of a non-constant element of U.

   (b) Show that the linear operator $\frac{d}{dx} : V \to V$ maps the subspace U to itself, i.e. $\frac{d}{dx}\,|_U \in L(U)$.

   (c) What are the eigenvectors of the linear operator $\frac{d}{dx} : V \to V$? (If $\frac{d}{dx}(f) = \lambda f$, with non-zero $f \in V$, then f is an eigenvector for $\frac{d}{dx}$, with eigenvalue λ.) Do any of them live in U?

   (d) Does U contain any eigenvectors of the linear operator $\left(\frac{d}{dx}\right)^2$?

   (e) Now let $W = C^\infty(\mathbb{R}, \mathbb{C})$ (the space of complex valued functions of a real variable, with derivatives of all orders). Let $X = \{f \in W : f(x + 2\pi) = f(x)\ \forall\, x \in \mathbb{R}\}$. What are the eigenvectors of $\frac{d}{dx} : X \to X$. What is their relationship with the functions you found in (d)?

22. Calculate $\int_0^{\pi/3} \sin x\, dx$ and $\int_0^{\pi/3} \cos x\, dx$. Without "doing any more integration", what is $\int_0^{\pi/3} 3\sin x + 4\cos x\, dx$? What property of integration are you using here?

23. Given $a, b \in \mathbb{R}$ with $a < b$, let $\theta_{[a,b]} : C(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ be given by $\theta_{[a,b]}(f) := \int_a^b f(x)\, dx$.

   (a) Prove that $\theta_{[a,b]}$ is linear (so belongs to the dual space $C(\mathbb{R}, \mathbb{R})^*$ of linear functions from $C(\mathbb{R}, \mathbb{R})$ to $\mathbb{R}$).

   (b) If $\int_0^1 f(x)\, dx = 1$, $\int_0^2 f(x)\, dx = 2$ and $\int_2^4 f(x)\, dx = 3$, what is $\int_1^4 f(x)\, dx$? What linear dependence relation between $\theta_{[0,1]}, \theta_{[0,2]}$, $\theta_{[1,4]}$ and $\theta_{[2,4]}$ are you using?

   (c) Are $\theta_{[0,1]}, \theta_{[0,2]}$ and $\theta_{[0,3]}$ linearly dependent?

   (d) Produce a function of the form $f(x) = a + bx + cx^2$ that fits (b).

24. Let V be an F-vector space, and $P \in L(V)$ a linear operator. If $P^2 = P$ then P is said to be a *projector*.

   (a) Show that if P is a projector, then so is $1 - P$ (where 1 is an abbreviation for the identity map).

   (b) Using $v = Pv + v - Pv$, show that $\mathrm{im}(P) + \mathrm{im}(1 - P) = V$, i.e. *every* element of V lies in this subspace.

   (c) By applying P to an element of $\mathrm{im}(1 - P)$, show that $\mathrm{im}(P) \cap \mathrm{im}(1 - P) = \{0\}$.

   (d) Deduce that $V \simeq \mathrm{im}(P) \oplus \mathrm{im}(1 - P)$ (see Questions 10 and 11 for the direct sum).

(e) Let $V = \mathbb{R}^2$, P the linear map represented by the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, i.e. $P = \ell_A$. Show that P is a projector. If $v = \begin{pmatrix} x \\ y \end{pmatrix}$, what are Pv and $(1 - P)v$?

(f) Give an example of a linear operator on $\mathbb{R}^2$ that is not a projector.

25. Let V be an F-vector space, and let U, W be subspaces of V. Recall that the map $f : U \oplus W \to U + W$ defined by $f((u, w)) := u + w$ is linear. Prove that $\ker(f) \simeq U \cap W$. What does the First Isomorphism Theorem now tell us? Deduce a relation between the dimensions of U, W, $U + W$ and $U \cap W$ (when U and V are finite-dimensional).

26. Let $V = \mathbb{R}^3$, and let U be the subspace Span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

(a) Describe U also as a null space, i.e. by an implicit equation. [Hint: what can you say about z?]

(b) Let $P_1 := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + U$ and $P_2 := \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + U$. Give examples of $v_1, v_2 \in \mathbb{R}^3$, different from those already used, such that $P_1 = v_1 + U$ and $P_2 = v_2 + U$.

(c) How would you describe P_1 and P_2 implicitly, as in (a), in terms of an equation for z? What about their sum, and what name might you give it?

(d) Give a natural geometrical description of the set V/U. [Hint: what kind of sets are P_1 and P_2?]

(e) What is the linear function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $U = \ker(f)$? (What is being set equal to 0?) What does the First Isomorphism Theorem say, as applied to f? Relate the isomorphism to your answer to (c).

27. (a) Given a prime number p, let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the finite field with p elements, and let $\theta : \mathbb{F}_p[x] \to \mathcal{F}(\mathbb{F}_p, \mathbb{F}_p)$ be the (linear) map obtained by sending a polynomial (the formal expression $a_0 + a_1x + \cdots + a_nx^n$) to the associated function, which takes an input $x \in \mathbb{F}_p$ and returns an output $a_0 + a_1x + \cdots + a_nx^n \in \mathbb{F}_p$. Find a non-zero element in the kernel of θ. [Hint: think of Fermat's Little Theorem from MAS114 (corollary to Theorem 7.10 in Semester 2).]

(b) How could we have predicted in advance that $\ker(\theta)$ would be non-zero? [Hint: look at the dimension of each side as a vector space over $\mathbb{F}_p$.]

(c) Prove that if $\mathbb{F}_p$ is replaced by $\mathbb{R}$ (so now $\theta : \mathbb{R}[x] \to \mathcal{F}(\mathbb{R}, \mathbb{R})$) then $\ker(\theta) = \{0\}$. Where does the argument employed in (b) break down? Is θ now an isomorphism?

28. Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 1 & 3 \\ 5 & 9 & 8 & 9 & 11 \end{pmatrix} \in M_{3,5}(\mathbb{R})$.

Convert A to reduced row echelon form, and find a basis for $\mathrm{Null}(A)$, i.e. the solution space to the set of homogeneous linear equations with coefficient matrix A. Check that $\dim(\mathrm{Null}(A)) + \mathrm{row\,rank}(A) = 5$.

29. The matrix $C \in M_2(\mathbb{R})$ has eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (eigenvalue 3) and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ (eigenvalue 2). Find C.

30. A linear transformation $\ell \in L(\mathbb{R}^3)$ is represented by the matrix $C = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ with respect to the standard basis of $\mathbb{R}^3$, i.e. $\ell = \ell_C$. Find the matrix A representing ℓ with respect to the basis $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

31. For a fixed field F and integer $n \geq 1$, consider a relation $\sim$ on the set $M_n(F)$, defined by $C \sim A$ if and only if there exists invertible $B \in M_n(F)$ (i.e. $B \in \mathrm{GL}_n(F)$) such that $C = BAB^{-1}$. Prove that $\sim$ is an equivalence relation. If we were to define a relation on the subset $\mathrm{GL}_n(F)$ of invertible matrices, using the same condition, what would we call it?

32. Let $\ell \in L(F^n)$, and let $A \in M_n(F)$ be the matrix representing ℓ with respect to a basis $\{\underline{b}_1, \ldots, \underline{b}_n\}$, the columns of a matrix $B \in \mathrm{GL}_n(F)$. Let $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, so $\underline{x} = \sum_{j=1}^n x_j \underline{e}_j$. Suppose that $\underline{x} = \sum_{j=1}^n x'_j \underline{b}_j$, so $\underline{x}' := \begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix}$ is the coordinate vector of $\underline{x}$ with respect to the basis $\{\underline{b}_1, \ldots, \underline{b}_n\}$.

Similarly let $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \underline{y} = \ell(\underline{x}) = \sum_{j=1}^n y'_j \underline{b}_j$, and $\underline{y}' := \begin{pmatrix} y'_1 \\ \vdots \\ y'_n \end{pmatrix}$, so $\underline{y}' = A\underline{x}'$. Prove the following.

(a) $\underline{x} = B\underline{x}'$.

(b) $\underline{x}' = B^{-1}\underline{x}$.

(c) $\underline{y}' = AB^{-1}\underline{x}$.

(d) $\underline{y} = BAB^{-1}\underline{x}$.

Deduce (reprove) that if C is the matrix representing ℓ with respect to the standard basis, then $C = BAB^{-1}$.

33. Let $\ell \in L(F^n)$, and let $A \in M_n(F)$ be the matrix representing ℓ with respect to a basis $\{\underline{b}_1, \ldots, \underline{b}_n\}$, the columns of a matrix $B \in \mathrm{GL}_n(F)$. Let $A' \in M_n(F)$ be the matrix representing ℓ with respect to a basis $\{\underline{b}'_1, \ldots, \underline{b}'_n\}$, the columns of a matrix $B' \in \mathrm{GL}_n(F)$. Prove that $A' = QAQ^{-1}$, where $Q = B'^{-1}B$. [Hint: consider, in two different ways, the matrix C representing ℓ with respect to the standard basis of F^n.]

34. For R a commutative ring, integer $n \geq 1$, and matrices $A, B \in M_n(R)$, prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, where the trace of A is defined by $\mathrm{tr}(A) := \sum_{i=1}^n A_{ii}$.

35. Let V be a vector space over $\mathbb{R}$, with inner product $\langle , \rangle$, so that the axioms IP1–IP4 are satisfied.

    (a) Prove carefully in full detail that $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$, for all $u, v, w \in V$, saying precisely which axiom you are using at each step.

    (b) Prove that $\langle u, \lambda v\rangle = \lambda\langle u, v\rangle$, for all $u, v \in V$ and $\lambda \in \mathbb{R}$.

    (c) Using only (a),(b) and the axioms IP1–IP4, prove that for any $u, v \in V$ and $t \in \mathbb{R}$,

    $$\langle tu + v, tu + v\rangle = t^2\langle u, u\rangle + 2t\langle u, v\rangle + \langle v, v\rangle.$$

36. Let V be a vector space over $\mathbb{R}$, with $\langle , \rangle : V \times V \to \mathbb{R}$ satisfying the axioms IP1–IP3 (but not necessarily IP4).

    (a) Prove that $\langle 0_V, 0_V\rangle = 0$.

    (b) For $V = \mathbb{R}^3$, give an example of $\langle , \rangle : V \times V \to \mathbb{R}$ satisfying the axioms IP1–IP3, and also $\langle v, v\rangle \geq 0$ for all $v \in \mathbb{R}^3$, but such that $\langle v, v\rangle = 0$ does not imply $v = \underline{0}$.

    (c) For $V = \mathbb{R}^3$, give an example of $\langle , \rangle : V \times V \to \mathbb{R}$ satisfying the axioms IP1–IP3, but not even satisfying $\langle v, v\rangle \geq 0 \ \forall\, v \in \mathbb{R}^3$.

37. If you have a protractor, draw the vectors $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ on squared paper, and measure the angle between them as accurately as you can. If you haven't got a protractor, guess the angle. Now calculate it precisely.

38. (For those doing MAS221 Analysis.) Let $V = C([a, b], \mathbb{R})$, with $\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$. Suppose that $g \in V$, with $g(x) \geq 0 \ \forall x \in [a, b]$. Suppose that $g(x_0) > 0$ for some $x_0 \in [a, b]$. Show that there exists an interval $I = [a, b] \cap (x_0 - \delta, x_0 + \delta)$ such that $g(x) \geq \epsilon\chi_I(x) \ \forall x \in [a, b]$, where $\epsilon = \frac{1}{2}g(x_0)$. (Recall the step functions in MAS221 Semester 2.) Using the appropriate proposition from MAS221, prove that $\int_a^b g(x)\,dx > 0$. Applying this to $g = f^2$, prove axiom IP4.

39. Let V be a real inner product space.

(a) Prove that $||\lambda v|| = |\lambda| \, ||v||$, for all $\lambda \in \mathbb{R}$, $v \in V$.

(b) Define $d : V \times V \to \mathbb{R}$ by $d(v, w) := ||v - w||$. Prove that

i. $d(v, w) = d(w, v), \;\; \forall v, w \in V$;

ii. $d(u, w) \leq d(u, v) + d(v, w) \;\; \forall u, v, w \in V$;

iii. $d(v, w) \geq 0 \;\; \forall v, w \in V$, with equality if and only if $v = w$.

(These are the axioms for a *metric space*.)

(c) Prove that if $v, w \in V$ then $||v+w||^2 = ||v||^2 + ||w||^2 \iff \langle v, w \rangle = 0$.

(d) Let W be a subspace, and for a fixed $v \in V$, suppose that $v = w + w'$, with $w \in W$ and $w' \in W^{\perp}$. Prove that w is the closest point of W to v, i.e. that if $u \in W$ then $d(v, u) \geq d(v, w)$, with equality if and only if $u = w$. [Hint: write $v - u = (v - w) + (w - u)$, and consider $||v - u||^2$.]

(e) Suppose that W is finite-dimensional, with orthogonal basis $\{w_1, \ldots, w_m\}$. For v and w as in the previous part, if $w = \sum_{i=1}^{m} \alpha_i w_i$, how would you calculate α_i?

40. (a) Find a basis for the subspace $P^{\perp}$ of $\mathbb{R}^3$, where P is the plane $x + 3y + 4z = 0$.

(b) Find a basis for the subspace $W^{\perp}$ of $\mathbb{R}^3$, where $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \right\}$.

(c) What is the relationship between $W, W^{\perp}, P$ and $P^{\perp}$?

41. Find the orthogonal projection of the vector $\underline{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ on the plane W : $x + 3y + 4z = 0$. Hence find the distance between $\underline{v}$ and W, i.e. of the point $(1, 1, 1)$ from this plane.

42. Apply the Gram-Schmidt process to the basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$, to get an orthogonal basis of $\mathbb{R}^3$. Check that it really is orthogonal. Now use it to produce an orthonormal basis.

43. Prove that if W is a subspace of a real inner product space V, then $W^{\perp} := \{v \in V \mid \langle v, w \rangle = 0 \;\; \forall w \in W\}$ is also a subspace.

44. (a) Continue the application of the Gram-Schmidt process to the subset $\{1, x, x^2, \ldots\}$ of $C([-1, 1], \mathbb{R})$, to produce the Legendre polynomials $P_4(x)$ and $P_5(x)$.

(b) Prove, by induction, that $P_n(x)$ is odd for odd n, even for even n.

(c) If $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} c_n P_n(x)$ for all $x \in [-1, 1]$ (without worrying too much about the correct analytic approach to the infinite sum), find c_0, c_1 and c_2.

45. Let V be a real inner product space, $T, S \in L(V)$. Suppose that T and S have adjoints T^* and S^*, respectively. Prove that $S + T$ has adjoint $S^* + T^*$, and that ST has adjoint T^*S^*. In the special case that V is finite-dimensional, what properties of matrices do these translate into?

46. Let V be a real inner product space, $T \in L(V)$,and suppose that $T_1^*, T_2^* \in L(V)$ are both adjoints for T, i.e. $\langle Tv, w\rangle = \langle v, T_1^* w\rangle = \langle v, T_2^* w\rangle$ for all $v, w \in V$. By considering $||T_1^* w - T_2^* w||^2$, prove that $T_1^* = T_2^*$, i.e. an adjoint, if it exists, is unique.

47. Let V be a complex inner product space, and suppose that $T \in L(V)$ has adjoint T^*. Prove that for any $\alpha \in \mathbb{C}$, αT has adjoint $\overline{\alpha} T^*$.

48. Let $z = x + iy, w = u + iv \in \mathbb{C}$, with $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\underline{u} = \begin{pmatrix} u \\ v \end{pmatrix}$. What is the relationship between the real inner product $\underline{x} \cdot \underline{u} = \underline{x}^t \underline{u}$ on $\mathbb{R}^2$ and the complex inner product $z\overline{w}$ on $\mathbb{C}$?

49. (a) Let $\{\underline{b}_1, \ldots, \underline{b}_n\}$ be a basis for $\mathbb{R}^n$, and $B = (\underline{b}_1 | \cdots | \underline{b}_n)$. Prove that $\{\underline{b}_1, \ldots, \underline{b}_n\}$ is an *orthonormal* basis if and only if B is an orthogonal matrix (i.e. $B^t B = I$).

    (b) Let $C \in M_n(\mathbb{R})$ with $C^t = C$, i.e. C is a real symmetric matrix. Let $f_C \in L(\mathbb{R}^n)$ be the linear operator $\underline{x} \mapsto C\underline{x}$ (so C represents f_C with respect to the standard basis of $\mathbb{R}^n$). Recall that, since f_C is self-adjoint, it has an orthonormal basis $\{\underline{b}_1, \ldots, \underline{b}_n\}$ of eigenvectors, say $C\underline{b}_i = \lambda_i \underline{b}_i$ for $1 \le i \le n$. Let $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Prove that if $B = (\underline{b}_i | \cdots | \underline{b}_n)$ then $D = B^t C B$.

    (c) With C and B as above, prove that if $\underline{x} = B\underline{X}$ then $\underline{x}^t C \underline{x} = \underline{X}^t D \underline{X}$. Applying this to $C = \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix}$, find X, Y such that $\frac{3}{2}x^2 - xy + \frac{3}{2}y^2 = X^2 + 2Y^2$. What is the curve $3x^2 - 2xy + 3y^2 = 4$? Sketch it in the x-y plane.

    (d) With C as in part (b), prove that if all $\lambda_i > 0$ then there exists an invertible matrix $P \in \mathrm{GL}_n(\mathbb{R})$ such that $C = P^t P$. [Hint: try $P = BFB^t$ for suitable F such that $F^t F = D$.]

    (e) Using the matrix P in the previous part, show that if all $\lambda_i > 0$ then $\langle , \rangle_C : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, defined by $\langle \underline{x}, \underline{y}\rangle_C := \underline{x}^t C \underline{y}$, is an inner product. Prove that if some $\lambda_i \le 0$ then $\langle , \rangle_C$ is not an inner product.

    (f) Find P in the special case that $C = \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix}$.

50. Find an orthogonal basis of eigenvectors for the Hermitian matrix $\begin{pmatrix} 1 & 1+i \\ 1-i & 1 \end{pmatrix}$. Check that they really are orthogonal.

51. Find an orthogonal basis of eigenvectors for the real symmetric matrix
$$A = \begin{pmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{pmatrix}.$$