

MAS220, Semester 2 Problems.

1. (a) Let $A, B \in M_n(\mathbb{R})$ be invertible. Prove that AB is invertible, with $(AB)^{-1} = B^{-1}A^{-1}$.
- (b) Let $A \in M_{m,n}(\mathbb{R}), B \in M_{n,p}(\mathbb{R})$. By considering the ij entry on each side, show that $(AB)^T = B^T A^T$.
- (c) Let $O_n := \{A \in M_n(\mathbb{R}) : A^T A = I\}$. Prove that O_n is a group under matrix multiplication. (It is called the orthogonal group.) [Hint: $A^T A = I$ shows that A is invertible, with $A^{-1} = A^T$, which you must show also belongs to O_n .]
- (d) What is O_1 ? Give examples of elements $A, B \in O_2$ such that $AB \neq BA$.
- (e) Prove that if $A \in O_n$ then $\det(A) = \pm 1$. Give examples of elements $A, B \in O_2$ with $\det(A) = 1$ and $\det(B) = -1$.
- (f) For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, let $\mathbf{x} \cdot \mathbf{y} := \mathbf{x}^T \mathbf{y}$ (ignoring the brackets on the 1-by-1 matrix, so $\mathbf{x} \cdot \mathbf{y} \in \mathbb{R}$). Prove that if $A \in O(n)$ then $(A\mathbf{x}) \cdot (A\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
- (g) Conversely, show that if $(A\mathbf{x}) \cdot (A\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then $A \in O(n)$. You might like to consider choosing $\mathbf{x} = e_i, \mathbf{y} = e_j$, where e_i is the vector with 1 in the i^{th} position and 0 elsewhere.
- (h) What has O_2 got to do with the group we called O_2 in Semester 1 (or in MAS114)?
2. (a) We define matrices $\mathbf{i}, \mathbf{j}, \mathbf{k} \in M_2(\mathbb{C})$ by

$$\mathbf{i} := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \mathbf{j} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathbf{k} := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Calculate $\mathbf{i}^2, \mathbf{j}^2, \mathbf{k}^2, \mathbf{ij}, \mathbf{ji}, \mathbf{j\mathbf{k}}, \mathbf{k\mathbf{j}}, \mathbf{k\mathbf{i}}$ and $\mathbf{i\mathbf{k}}$.

- (b) Define a ring homomorphism $\theta : \mathbb{H} \rightarrow M_2(\mathbb{C})$, where $\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$ is Hamilton's quaternion ring. (You must say what θ is, but you need not check in detail that it is a ring homomorphism.) Is θ injective? Is it surjective?
- (c) Given $\alpha = a + bi + cj + dk \in \mathbb{H}$, what is $\det(\theta(\alpha))$?
3. Let $P = \text{Span} \left\{ \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\} = \left\{ \lambda \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} : \lambda, \mu \in \mathbb{R} \right\}$, the unique plane through the origin containing the points $(5, -1, -1)$ and $(1, 1, -1)$. Express P in the form $ax + by + cz = 0$. [Hint: find a set of linear equations for a, b, c , and use row reduction to solve them.]
4. (deleted)

5. Let V be an F -vector space, and let U, W be subspaces of V . We define the *sum*

$$U + W := \{u + w : u \in U, w \in W\}.$$

- (a) Prove that $U + W$ is a subspace of V .

- (b) Let $V = \mathbb{R}^4$, with subspaces $U = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ and $W =$

$\text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$. What are $\dim(U)$ and $\dim(W)$? (The dimension of a vector space is the number of elements in a basis.) Show

that $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} \in U + W$, and find two different expressions $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} = u + w$,

with $u \in U, w \in W$. Find some $v \in V$ with $v \notin U + W$. What is $\dim(U + W)$ in this example?

- (c) Now find 2-dimensional subspaces U and W of \mathbb{R}^4 such that $U + W = \mathbb{R}^4$.

6. Let V be an F -vector space, and let U, W be subspaces of V .

- (a) Prove that $U \cap W$ is a subspace of V .

- (b) Let $V = \mathbb{R}^6$. Find 3-dimensional subspaces U_i, W_i for $1 \leq i \leq 4$ such that

$$\dim(U_1 + W_1) = 3, \dim(U_2 + W_2) = 4, \dim(U_3 + W_3) = 5, \dim(U_4 + W_4) = 6.$$

[Hint: keep it simple. Try spans of subsets of the standard basis for \mathbb{R}^6 .]

- (c) In each of your examples, what is $\dim(U_i \cap W_i)$?

- (d) In general, can you guess a relationship between the dimensions of the subspaces $U, W, U + W$ and $U \cap W$ (assuming they are finite-dimensional)?

7. (a) Let $v_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $v_4 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Why must

this subset of \mathbb{R}^3 be linearly dependent? Find all the possible linear dependence relations $x_1 v_1 + x_2 v_2 + x_3 v_3 + x_4 v_4 = 0$. [Hint: the possible

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ form the null space of a certain matrix A , which you should

row reduce to solve for x_1, x_2, x_3 in terms of x_4 .] Hence express each v_i as a linear combination of the others. (Don't forget this bit!)

- (b) Now, let $v_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $v_4 = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$. Again, find all the possible linear dependence relations $x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 = 0$. Is it still the case that each v_i can be expressed as a linear combination of the others?

8. Let $L = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$. Express this line through the origin in \mathbb{R}^3 also as $\text{Null}(A)$ for some matrix $A \in M_{2,3}(\mathbb{R})$. [Hint: look at just one row $[a, b, c]$ of A , and find all the possibilities for $[a, b, c]$ in order to produce more than one row.] Now find a second, completely different, pair of homogeneous linear equations describing the same line.

9. Let C be the subspace $\text{Null}(H)$ of \mathbb{F}_2^7 , where $H = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$.

- (a) Find a basis for C . (You may write the vectors as horizontal strings.) What is its dimension? Could you have predicted this in advance?
- (b) Putting these basis strings in the rows of a matrix G , and considering $\mathbf{x}G$ for all possible $\mathbf{x} \in (\mathbb{F}_2)_4$, list all the elements of C . (You could construct a table of \mathbf{x} and $\mathbf{x}G$.) What is the smallest number of 1s in a non-zero element of C ? Call this number d .
- (c) How many pairs $c_1, c_2 \in C$ with $c_1 \neq c_2$ are there? Without checking them all, prove that such c_1 and c_2 must differ in at least d places. [Hint: consider their difference.]
- (d) If $v = 1000101$, show that $v \notin C$. Find an element c of C that differs from v in precisely one place. Why must c be unique with this property? How does Hv tell you at which place c and v differ?
- (e) If a string of 0s and 1s is broken up into blocks \mathbf{x} of length 4, each \mathbf{x} can be replaced by an element $c = \mathbf{x}G$ of C , now of length 7. Now if an error occurs in c , at one place, producing some v , we can correct the error by finding the unique element of C differing from v in only one place, thus turning v back into c . Then \mathbf{x} can be recovered from c . For the c in the previous part, find \mathbf{x} .

10. Let V, W be finite-dimensional vector spaces over a field F , with $\dim(V) = n$, $\dim(W) = m$. We define the *direct sum*

$$V \oplus W := \{(v, w) : v \in V, w \in W\}.$$

Note that as a set this is the same as the cartesian product $V \times W$, seen in MAS110. We define operations of addition and scalar multiplication on

$V \oplus W$ by

$$(v_1, w_1) + (v_2, w_2) := (v_1 + v_2, w_1 + w_2) \quad \forall v_1, v_2 \in V, w_1, w_2 \in W;$$

$$\lambda(v, w) := (\lambda v, \lambda w) \quad \forall \lambda \in F, v \in V, w \in W.$$

- (a) Prove carefully that with these operations, $V \oplus W$ is a vector space.
- (b) Prove that if $\{v_1, v_2, \dots, v_n\}$ is a basis for V and $\{w_1, \dots, w_m\}$ is a basis for W , then $\{(v_1, 0), \dots, (v_n, 0), (0, w_1), \dots, (0, w_m)\}$ is a basis for $V \oplus W$. What then is $\dim(V \oplus W)$?
11. Let V be a vector space over a field F , and let U, W be subspaces of V .
- (a) Show that the map $f : U \oplus W \rightarrow U + W$ defined by $f((u, w)) := u + w$ is linear.
- (b) Suppose that $f((u_1, w_1)) = f((u_2, w_2))$. Prove that $u_1 - u_2 \in U \cap W$.
- (c) Deduce that if $U \cap W = \{0\}$ then f is an isomorphism of vector spaces and $\dim(U + W) = \dim(U) + \dim(W)$ (if U and W are finite-dimensional).
- (d) Prove that if $\dim(U) + \dim(W) > \dim(V)$ then $U \cap W \neq \{0\}$.
- (e) In \mathbb{R}^3 , let $U = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ and $W = \text{Span} \left\{ \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$. Why must $\dim(U \cap W) = 1$? Find $v \in \mathbb{R}^3$ such that $U \cap W = \text{Span}\{v\}$.
12. Let P be the plane $x + 3y + 4z = 0$ in \mathbb{R}^3 .
- (a) Of what matrix is P the null-space, i.e. P is the solution space of a system of homogeneous linear equations with what coefficient matrix?
- (b) Find vectors $v_1, v_2 \in \mathbb{R}^3$ such that $P = \text{Span}\{v_1, v_2\}$.
- (c) If $f : P \rightarrow \mathbb{R}$ is a linear function (i.e. a linear map between these \mathbb{R} -vector spaces), prove that for some $a, b \in \mathbb{R}$, $f(sv_1 + tv_2) = as + bt$.
- (d) Using $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = sv_1 + tv_2$, with the v_1, v_2 you found in (b), substitute for x, y and z in terms of s and t , and hence show that the linear functions $x - 3y + 2z$ and $3x + 3y + 10z$ are the same when restricted to P . How else could we have seen this? [Hint: consider the difference of these functions.]
- (e) Evaluate the functions $x - 3y + 2z$ and $3x + 3y + 10z$ at the point $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. Now express $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ in the form $sv_1 + tv_2$, hence evaluate the same functions in a different way at this point, using what you found in (d).

- (f) Find all possible $[a, b, c] \in \mathbb{R}_3$ such that the restriction to P of the linear function $ax + by + cz$ on \mathbb{R}^3 is the linear function $2s + 3t$ on P . Would the same method work for any linear function of the form $ds + et$, e.g. $-s + 2t$?
13. Let $f : V \rightarrow W$ be a linear map. Prove carefully that if f is a bijection then the inverse map $f^{-1} : W \rightarrow V$ is also linear. [Hint: given $w \in W$, write it as $f(v)$ for some $v \in V$.]
14. Let V be a vector space over a field F . Given any fixed $\mu \in F$, let $\theta_\mu : V \rightarrow V$ be the map $v \mapsto \mu v$. Prove that θ_μ is a linear map, justifying each step carefully. Prove also that if $\dim(V) = 1$ then every linear map $f : V \rightarrow V$ is necessarily of the form θ_μ , for some μ . [Hint: let $\{v\}$ be a basis for V . Now $f(v) \in V$, but $\{v\}$ is a basis for V , so $f(v) = \mu v = \theta_\mu(v)$ for some $\mu \in F$. You must show that with the same μ , $f(w) = \mu w$ for any $w \in V$, not just for $w = v$.]
15. While \mathbb{R}^2 is 2-dimensional as an \mathbb{R} -vector space (i.e. $\dim_{\mathbb{R}}(\mathbb{R}^2) = 2$), via the usual bijection $\mathbb{R}^2 \simeq \mathbb{C}$ ($\begin{pmatrix} a \\ b \end{pmatrix} \mapsto a + ib$) we may view it as the 1-dimensional \mathbb{C} -vector space \mathbb{C} ($\dim_{\mathbb{C}}(\mathbb{R}^2) = \dim_{\mathbb{C}}(\mathbb{C}) = 1$).
- (a) Prove that the \mathbb{R} -linear map $\text{rot}_\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, represented by the matrix $\begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$, is also \mathbb{C} -linear as a map from \mathbb{C} to \mathbb{C} . [Hint: Referring to the previous question, what is μ , i.e. what complex number do you multiply by to effect this rotation? You might like to recall the polar form of a complex number.]
- (b) Find two linearly independent eigenvectors for the matrix $\begin{pmatrix} \cos \beta & \sin \beta \\ \sin \beta & -\cos \beta \end{pmatrix}$ representing the \mathbb{R} -linear map $\text{ref}_\beta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. [Hint: $\beta = 2(\beta/2)$.] Is ref_β also \mathbb{C} -linear? In terms of complex numbers, how else might you describe ref_0 ?
- (c) What are all the possible bases for the 1-dimensional \mathbb{C} -vector space \mathbb{C} ?
16. (deleted)
17. (a) If F is any field and X is a finite non-empty set, what is the dimension of the F -vector space $\mathcal{F}(X, F)$ of all functions from X to F ? [Hint: what happened when $X = \{1, 2, \dots, n\}$?]
- (b) More generally, if X is a finite non-empty set and V is a finite-dimensional vector space over F , with $\dim(V) = m$, what is $\dim(\mathcal{F}(X, V))$?
18. Let V, W be vector spaces over a field F , and let $V^* = L(V, F)$ and $W^* = L(W, F)$ be the dual spaces (of linear functions on V and on W). Let $\theta : V \rightarrow W$ be a linear map. We define a map $\theta^* : W^* \rightarrow V^*$ by

$$\theta^*(g) := g \circ \theta, \quad \forall g \in W^*,$$

i.e. $(\theta^*(g))(v) := g(\theta(v))$, $\forall g \in W^*, v \in V$. (You might like to draw a diagram to see what is going on.) Since a composition of linear maps is linear, we know that $\theta^*(g)$ really does belong to V^* , i.e. that $g \circ \theta$ is a linear map from V to F .

- (a) Prove that $\theta^* : W^* \rightarrow V^*$ is linear, i.e. that $\theta^*(g_1 + g_2) = \theta^*(g_1) + \theta^*(g_2)$ and $\theta^*(\lambda g) = \lambda \theta^*(g)$ (\forall etc.).
- (b) In the case that $V = F^n$ and $W = F^m$, with θ given by the matrix $A \in M_{m,n}(F)$ (i.e. $\theta = \ell_A$), show that if $g \in W^*$ is the linear function specified by $\mathbf{a} \in F_m$ (i.e. $g = \ell_{\mathbf{a}}$, so $g(\mathbf{x}) = \mathbf{a}\mathbf{x} = \sum_{j=1}^m a_j x_j$), then $\theta^*(g)$ is associated with $\mathbf{a}A \in F_n$ (i.e. $\theta^*(g) = \ell_{\mathbf{a}A}$).
- (c) Note that if V is a subspace of W and θ is the inclusion of V in W , then θ^* is just the restriction of linear functions from W to the subspace V . We had an example of this in Question 12, with $W = \mathbb{R}^3$ and $V = P$, the plane $x + 3y + 4z = 0$. Since $\begin{pmatrix} s \\ t \end{pmatrix} \mapsto sv_1 + tv_2$ is an isomorphism of \mathbb{R}^2 with P , we really have $\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, $\theta \left(\begin{pmatrix} s \\ t \end{pmatrix} \right) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Using the v_1, v_2 you found in Question 12, give the matrix A such that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \begin{pmatrix} s \\ t \end{pmatrix}$. In Question 12(d) you calculated the restriction to P of the linear functions $x - 3y + 2z$ and $3x + 3y + 10z$. Recover the same answers using (b).

19. Let $D := x \frac{d}{dx}$.

- (a) First look at D as a linear operator on $\mathbb{R}[x]$. What are its eigenvalues and eigenvectors? (If $\theta \in L(V)$, and if $\theta(v) = \lambda v$, with $0 \neq v \in V$, then by definition v is an eigenvector for θ , with eigenvalue λ .)
- (b) Calculate an expression for D^2 in terms of x and $\frac{d}{dx}$, and check directly that it does what you would expect to the the functions you found in (a).
- (c) Now look at D as a linear operator on $C^\infty(\mathbb{R}, \mathbb{R})$. Show that there is no eigenvector with eigenvalue $\lambda = -1$ or $\lambda = 1/2$. What if we restrict to $C^\infty([1, 2], \mathbb{R})$?
- (d) Find, in $C^1(\mathbb{R}, \mathbb{R})$, an eigenvector for D , with eigenvalue $\lambda = 3$, that you did not already have in the subspace $\mathbb{R}[x]$.

20. (a) Prove that the operator $D = x \frac{d}{dx}$ on $C^\infty(\mathbb{R}, \mathbb{R})$ (i.e. $D(y) = x \frac{dy}{dx}$) is linear.

- (b) What goes wrong if you try to prove that the operator $D' : y \mapsto y \frac{dy}{dx}$ on $C^\infty(\mathbb{R}, \mathbb{R})$ is linear?

21. Let $V = C^\infty(\mathbb{R}, \mathbb{R})$, and let $U = \{f \in V : f(x + 2\pi) = f(x) \forall x \in \mathbb{R}\}$.
- Show that U is a subspace of V . Give an example of a non-constant element of U .
 - Show that the linear operator $\frac{d}{dx} : V \rightarrow V$ maps the subspace U to itself, i.e. $\frac{d}{dx}|_U \in L(U)$.
 - What are the eigenvectors of the linear operator $\frac{d}{dx} : V \rightarrow V$? (If $\frac{d}{dx}(f) = \lambda f$, with non-zero $f \in V$, then f is an eigenvector for $\frac{d}{dx}$, with eigenvalue λ .) Do any of them live in U ?
 - Does U contain any eigenvectors of the linear operator $(\frac{d}{dx})^2$?
 - Now let $W = C^\infty(\mathbb{R}, \mathbb{C})$ (the space of complex valued functions of a real variable, with derivatives of all orders). Let $X = \{f \in W : f(x + 2\pi) = f(x) \forall x \in \mathbb{R}\}$. What are the eigenvectors of $\frac{d}{dx} : X \rightarrow X$. What is their relationship with the functions you found in (d)?
22. Calculate $\int_0^{\pi/3} \sin x \, dx$ and $\int_0^{\pi/3} \cos x \, dx$. Without “doing any more integration”, what is $\int_0^{\pi/3} 3 \sin x + 4 \cos x \, dx$? What property of integration are you using here?
23. Given $a, b \in \mathbb{R}$ with $a < b$, let $\theta_{[a,b]} : C(\mathbb{R}, \mathbb{R}) \rightarrow \mathbb{R}$ be given by $\theta_{[a,b]}(f) := \int_a^b f(x) \, dx$.
- Prove that $\theta_{[a,b]}$ is linear (so belongs to the dual space $C(\mathbb{R}, \mathbb{R})^*$ of linear functions from $C(\mathbb{R}, \mathbb{R})$ to \mathbb{R}).
 - If $\int_0^1 f(x) \, dx = 1$, $\int_0^2 f(x) \, dx = 2$ and $\int_2^4 f(x) \, dx = 3$, what is $\int_1^4 f(x) \, dx$? What linear dependence relation between $\theta_{[0,1]}$, $\theta_{[0,2]}$, $\theta_{[1,4]}$ and $\theta_{[2,4]}$ are you using?
 - Are $\theta_{[0,1]}$, $\theta_{[0,2]}$ and $\theta_{[0,3]}$ linearly dependent?
 - Produce a function of the form $f(x) = a + bx + cx^2$ that fits (b).
24. Let V be an F -vector space, and $P \in L(V)$ a linear operator. If $P^2 = P$ then P is said to be a *projector*.
- Show that if P is a projector, then so is $1 - P$ (where 1 is an abbreviation for the identity map).
 - Using $v = Pv + v - Pv$, show that $\text{im}(P) + \text{im}(1 - P) = V$, i.e. every element of V lies in this subspace.
 - By applying P to an element of $\text{im}(1 - P)$, show that $\text{im}(P) \cap \text{im}(1 - P) = \{0\}$.
 - Deduce that $V \simeq \text{im}(P) \oplus \text{im}(1 - P)$ (see Questions 10 and 11 for the direct sum).

- (e) Let $V = \mathbb{R}^2$, P the linear map represented by the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, i.e. $P = \ell_A$. Show that P is a projector. If $v = \begin{pmatrix} x \\ y \end{pmatrix}$, what are Pv and $(1 - P)v$?
- (f) Give an example of a linear operator on \mathbb{R}^2 that is not a projector.
25. Let V be an F -vector space, and let U, W be subspaces of V . Recall that the map $f : U \oplus W \rightarrow U + W$ defined by $f((u, w)) := u + w$ is linear. Prove that $\ker(f) \simeq U \cap W$. What does the First Isomorphism Theorem now tell us? Deduce a relation between the dimensions of U , W , $U + W$ and $U \cap W$ (when U and W are finite-dimensional).
26. Let $V = \mathbb{R}^3$, and let U be the subspace $\text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.
- (a) Describe U also as a null space, i.e. by an implicit equation. [Hint: what can you say about z ?]
- (b) Let $P_1 := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + U$ and $P_2 := \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + U$. Give examples of $v_1, v_2 \in \mathbb{R}^3$, different from those already used, such that $P_1 = v_1 + U$ and $P_2 = v_2 + U$.
- (c) How would you describe P_1 and P_2 implicitly, as in (a), in terms of an equation for z ? What about their sum, and what name might you give it?
- (d) Give a natural geometrical description of the set V/U . [Hint: what kind of sets are P_1 and P_2 ?]
- (e) What is the linear function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that $U = \ker(f)$? (What is being set equal to 0?) What does the First Isomorphism Theorem say, as applied to f ? Relate the isomorphism to your answer to (c).
27. (a) Given a prime number p , let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the finite field with p elements, and let $\theta : \mathbb{F}_p[x] \rightarrow \mathcal{F}(\mathbb{F}_p, \mathbb{F}_p)$ be the (linear) map obtained by sending a polynomial (the formal expression $a_0 + a_1x + \cdots + a_nx^n$) to the associated function, which takes an input $x \in \mathbb{F}_p$ and returns an output $a_0 + a_1x + \cdots + a_nx^n \in \mathbb{F}_p$. Find a non-zero element in the kernel of θ . [Hint: think of Fermat's Little Theorem from MAS114 (corollary to Theorem 7.10 in Semester 2).]
- (b) How could we have predicted in advance that $\ker(\theta)$ would be non-zero? [Hint: look at the dimension of each side as a vector space over \mathbb{F}_p .]
- (c) Prove that if \mathbb{F}_p is replaced by \mathbb{R} (so now $\theta : \mathbb{R}[x] \rightarrow \mathcal{F}(\mathbb{R}, \mathbb{R})$) then $\ker(\theta) = \{0\}$. Where does the argument employed in (b) break down? Is θ now an isomorphism?

28. Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 1 & 3 \\ 5 & 9 & 8 & 9 & 11 \end{pmatrix} \in M_{3,5}(\mathbb{R})$.

Convert A to reduced row echelon form, and find a basis for $\text{Null}(A)$, i.e. the solution space to the set of homogeneous linear equations with coefficient matrix A . Check that $\dim(\text{Null}(A)) + \text{row rank}(A) = 5$.

29. The matrix $C \in M_2(\mathbb{R})$ has eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (eigenvalue 3) and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ (eigenvalue 2). Find C . Hint: let $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. What is A ?

30. A linear transformation $\ell \in L(\mathbb{R}^3)$ is represented by the matrix $C = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ with respect to the standard basis of \mathbb{R}^3 , i.e. $\ell = \ell_C$. Find the

matrix A representing ℓ with respect to the basis $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

31. For a fixed field F and integer $n \geq 1$, consider a relation \sim on the set $M_n(F)$, defined by $C \sim A$ if and only if there exists invertible $B \in M_n(F)$ (i.e. $B \in \text{GL}_n(F)$) such that $C = BAB^{-1}$. Prove that \sim is an equivalence relation. If we were to define a relation on the subset $\text{GL}_n(F)$ of invertible matrices, using the same condition, what would we call it?

32. Let $\ell \in L(F^n)$, and let $A \in M_n(F)$ be the matrix representing ℓ with respect to a basis $\{\underline{b}_1, \dots, \underline{b}_n\}$, the columns of a matrix $B \in \text{GL}_n(F)$. Let

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \text{ so } \underline{x} = \sum_{j=1}^n x_j \underline{e}_j. \text{ Suppose that } \underline{x} = \sum_{j=1}^n x'_j \underline{b}_j, \text{ so } \underline{x}' :=$$

$\begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix}$ is the coordinate vector of \underline{x} with respect to the basis $\{\underline{b}_1, \dots, \underline{b}_n\}$.

Similarly let $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \underline{y} = \ell(\underline{x}) = \sum_{j=1}^n y'_j \underline{b}_j$, and $\underline{y}' := \begin{pmatrix} y'_1 \\ \vdots \\ y'_n \end{pmatrix}$, so $\underline{y}' =$

$A\underline{x}'$. Prove the following.

- $\underline{x} = B\underline{x}'$.
- $\underline{x}' = B^{-1}\underline{x}$.
- $\underline{y}' = AB^{-1}\underline{x}$.
- $\underline{y} = BAB^{-1}\underline{x}$.

Deduce (reprove) that if C is the matrix representing ℓ with respect to the standard basis, then $C = BAB^{-1}$.

33. Let $\ell \in L(F^n)$, and let $A \in M_n(F)$ be the matrix representing ℓ with respect to a basis $\{b_1, \dots, b_n\}$, the columns of a matrix $B \in GL_n(F)$. Let $A' \in M_n(F)$ be the matrix representing ℓ with respect to a basis $\{b'_1, \dots, b'_n\}$, the columns of a matrix $B' \in GL_n(F)$. Prove that $A' = QAQ^{-1}$, where $Q = B'^{-1}B$. [Hint: consider, in two different ways, the matrix C representing ℓ with respect to the standard basis of F^n .]
34. For R a commutative ring, integer $n \geq 1$, and matrices $A, B \in M_n(R)$, prove that $\text{tr}(AB) = \text{tr}(BA)$, where the trace of A is defined by $\text{tr}(A) := \sum_{i=1}^n A_{ii}$.
35. Let V be a vector space over \mathbb{R} , with inner product $\langle \cdot, \cdot \rangle$, so that the axioms IP1–IP4 are satisfied.
- Prove carefully in full detail that $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$, for all $u, v, w \in V$, saying precisely which axiom you are using at each step.
 - Prove that $\langle u, \lambda v \rangle = \lambda \langle u, v \rangle$, for all $u, v \in V$ and $\lambda \in \mathbb{R}$.
 - Using only (a),(b) and the axioms IP1–IP4, prove that for any $u, v \in V$ and $t \in \mathbb{R}$,

$$\langle tu + v, tu + v \rangle = t^2 \langle u, u \rangle + 2t \langle u, v \rangle + \langle v, v \rangle.$$

36. Let V be a vector space over \mathbb{R} , with $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ satisfying the axioms IP1–IP3 (but not necessarily IP4).
- Prove that $\langle 0_V, 0_V \rangle = 0$.
 - For $V = \mathbb{R}^3$, give an example of $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ satisfying the axioms IP1–IP3, and also $\langle v, v \rangle \geq 0$ for all $v \in \mathbb{R}^3$, but such that $\langle v, v \rangle = 0$ does not imply $v = \underline{0}$.
 - For $V = \mathbb{R}^3$, give an example of $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ satisfying the axioms IP1–IP3, but not even satisfying $\langle v, v \rangle \geq 0 \forall v \in \mathbb{R}^3$.
37. If you have a protractor, draw the vectors $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ on squared paper, and measure the angle between them as accurately as you can. If you haven't got a protractor, guess the angle. Now calculate it precisely.
38. (For those doing MAS221 Analysis.) Let $V = C([a, b], \mathbb{R})$, with $\langle f, g \rangle = \int_a^b f(x)g(x) dx$. Suppose that $g \in V$, with $g(x) \geq 0 \forall x \in [a, b]$. Suppose that $g(x_0) > 0$ for some $x_0 \in [a, b]$. Show that there exists an interval $I = [a, b] \cap (x_0 - \delta, x_0 + \delta)$ such that $g(x) \geq \epsilon \chi_I(x) \forall x \in [a, b]$, where $\epsilon = \frac{1}{2}g(x_0)$. (Recall the step functions in MAS221 Semester 2.) Using the appropriate proposition from MAS221, prove that $\int_a^b g(x) dx > 0$. Applying this to $g = f^2$, prove axiom IP4.

39. Let V be a real inner product space.
- Prove that $\|\lambda v\| = |\lambda| \|v\|$, for all $\lambda \in \mathbb{R}$, $v \in V$.
 - Define $d : V \times V \rightarrow \mathbb{R}$ by $d(v, w) := \|v - w\|$. Prove that
 - $d(v, w) = d(w, v)$, $\forall v, w \in V$;
 - $d(u, w) \leq d(u, v) + d(v, w)$ $\forall u, v, w \in V$;
 - $d(v, w) \geq 0$ $\forall v, w \in V$, with equality if and only if $v = w$.
 (These are the axioms for a *metric space*.)
 - Prove that if $v, w \in V$ then $\|v+w\|^2 = \|v\|^2 + \|w\|^2 \iff \langle v, w \rangle = 0$.
 - Let W be a subspace, and for a fixed $v \in V$, suppose that $v = w + w'$, with $w \in W$ and $w' \in W^\perp$. Prove that w is the closest point of W to v , i.e. that if $u \in W$ then $d(v, u) \geq d(v, w)$, with equality if and only if $u = w$. [Hint: write $v - u = (v - w) + (w - u)$, and consider $\|v - u\|^2$.]
 - Suppose that W is finite-dimensional, with orthogonal basis $\{w_1, \dots, w_m\}$. For v and w as in the previous part, if $w = \sum_{i=1}^m \alpha_i w_i$, how would you calculate α_i ?
40. (a) Find a basis for the subspace P^\perp of \mathbb{R}^3 , where P is the plane $x + 3y + 4z = 0$.
- (b) Find a basis for the subspace W^\perp of \mathbb{R}^3 , where $W = \text{Span} \left\{ \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \right\}$.
- (c) What is the relationship between W, W^\perp, P and P^\perp ?
41. Find the orthogonal projection of the vector $\underline{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ on the plane $W : x + 3y + 4z = 0$. Hence find the distance between \underline{v} and W , i.e. of the point $(1, 1, 1)$ from this plane.
42. Apply the Gram-Schmidt process to the basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$, to get an orthogonal basis of \mathbb{R}^3 . Check that it really is orthogonal. Now use it to produce an orthonormal basis.
43. Prove that if W is a subspace of a real inner product space V , then $W^\perp := \{v \in V \mid \langle v, w \rangle = 0 \ \forall w \in W\}$ is also a subspace.
44. (a) Continue the application of the Gram-Schmidt process to the subset $\{1, x, x^2, \dots\}$ of $C([-1, 1], \mathbb{R})$, to produce the Legendre polynomials $P_4(x)$ and $P_5(x)$.
- (b) Prove, by induction, that $P_n(x)$ is odd for odd n , even for even n .

- (c) If $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} c_n P_n(x)$ for all $x \in [-1, 1]$ (without worrying too much about the correct analytic approach to the infinite sum), find c_0, c_1 and c_2 .
45. Let V be a real inner product space, $T, S \in L(V)$. Suppose that T and S have adjoints T^* and S^* , respectively. Prove that $S + T$ has adjoint $S^* + T^*$, and that ST has adjoint T^*S^* . In the special case that V is finite-dimensional, what properties of matrices do these translate into?
46. Let V be a real inner product space, $T \in L(V)$, and suppose that $T_1^*, T_2^* \in L(V)$ are both adjoints for T , i.e. $\langle Tv, w \rangle = \langle v, T_1^*w \rangle = \langle v, T_2^*w \rangle$ for all $v, w \in V$. By considering $\|T_1^*w - T_2^*w\|^2$, prove that $T_1^* = T_2^*$, i.e. an adjoint, if it exists, is unique.
47. Let V be a complex inner product space, and suppose that $T \in L(V)$ has adjoint T^* . Prove that for any $\alpha \in \mathbb{C}$, αT has adjoint $\bar{\alpha}T^*$.
48. Let $z = x + iy, w = u + iv \in \mathbb{C}$, with $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \underline{u} = \begin{pmatrix} u \\ v \end{pmatrix}$. What is the relationship between the real inner product $\underline{x} \cdot \underline{u} = \underline{x}^t \underline{u}$ on \mathbb{R}^2 and the complex inner product $z\bar{w}$ on \mathbb{C} ?
49. (a) Let $\{\underline{b}_1, \dots, \underline{b}_n\}$ be a basis for \mathbb{R}^n , and $B = (\underline{b}_1 | \dots | \underline{b}_n)$. Prove that $\{\underline{b}_1, \dots, \underline{b}_n\}$ is an *orthonormal* basis if and only if B is an orthogonal matrix (i.e. $B^t B = I$).
- (b) Let $C \in M_n(\mathbb{R})$ with $C^t = C$, i.e. C is a real symmetric matrix. Let $f_C \in L(\mathbb{R}^n)$ be the linear operator $\underline{x} \mapsto C\underline{x}$ (so C represents f_C with respect to the standard basis of \mathbb{R}^n). Recall that, since f_C is self-adjoint, it has an orthonormal basis $\{\underline{b}_1, \dots, \underline{b}_n\}$ of eigenvectors, say $C\underline{b}_i = \lambda_i \underline{b}_i$ for $1 \leq i \leq n$. Let $D = \text{diag}(\lambda_1, \dots, \lambda_n)$. Prove that if $B = (\underline{b}_1 | \dots | \underline{b}_n)$ then $D = B^t C B$.
- (c) With C and B as above, prove that if $\underline{x} = B\underline{X}$ then $\underline{x}^t C \underline{x} = \underline{X}^t D \underline{X}$. Applying this to $C = \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix}$, find X, Y such that $\frac{3}{2}x^2 - xy + \frac{3}{2}y^2 = X^2 + 2Y^2$. What is the curve $3x^2 - 2xy + 3y^2 = 4$? Sketch it in the x - y plane.
- (d) With C as in part (b), prove that if all $\lambda_i > 0$ then there exists an invertible matrix $P \in \text{GL}_n(\mathbb{R})$ such that $C = P^t P$. [Hint: try $P = BFB^t$ for suitable F such that $F^t F = D$.]
- (e) Using the matrix P in the previous part, show that if all $\lambda_i > 0$ then $\langle \cdot, \cdot \rangle_C : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, defined by $\langle \underline{x}, \underline{y} \rangle_C := \underline{x}^t C \underline{y}$, is an inner product. Prove that if some $\lambda_i \leq 0$ then $\langle \cdot, \cdot \rangle_C$ is not an inner product.
- (f) Find P in the special case that $C = \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix}$.

50. Find an orthogonal basis of eigenvectors for the Hermitian matrix $\begin{pmatrix} 1 & 1+i \\ 1-i & 1 \end{pmatrix}$.
Check that they really are orthogonal.

51. Find an orthogonal basis of eigenvectors for the real symmetric matrix
 $A = \begin{pmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{pmatrix}$.