The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS

Algebra **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Sample exam. Total marks 60.

1 (i) What does it mean for a function $f : G \to H$, where G and H are groups, to be a *group homomorphism*? ***(1 mark)***

(ii) Given a group G, and a subgroup N of G, what does it mean for N to be *normal*? ***(1 mark)***

(iii) Given a group homomorphism $f : G \to H$, define the *kernel,* $\ker(f)$, and prove that it is a normal subgroup of G. You may assume that $f(e_G) = e_H$ (where e_G and e_H are the neutral elements of G and H, respectively), and that $f(g^{-1}) = (f(g))^{-1}$ for all $g \in G$. ***(4 marks)***

(iv) In the group S_3 of permutations of $\{1, 2, 3\}$, is the cyclic subgroup $\langle(1\,2\,3)\rangle$ generated by the 3-cycle $(1\,2\,3)$ a normal subgroup? In the group S_4 of permutations of $\{1, 2, 3, 4\}$, is the cyclic subgroup $\langle(1\,2\,3\,4)\rangle$ generated by the 4-cycle $(1\,2\,3\,4)$ a normal subgroup? You need only justify your answer to the second of these two questions. ***(2 marks)***

2 (i) Prove carefully that if R is a commutative ring and $a, b \in R$, then $(ab)^2 = a^2b^2$. ***(1 mark)***

(ii) Prove that if R is a commutative ring in which $2 = 0$, i.e. $1 + 1 = 0$, then $(a + b)^2 = a^2 + b^2$, for any $a, b \in R$. You may assume that $0.r = 0$ for any $r \in R$. ***(1 mark)***

(iii) Let $R = M_2(\mathbb{F}_2)$, the ring of 2-by-2 matrices with entries in the finite field $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. What is the cardinality $|R|$? Find elements $A, B \in R$ such that $(AB)^2 \neq A^2B^2$ and $(A + B)^2 \neq A^2 + B^2$. ***(4 marks)***

(iv) Prove that there is no element $a \in \mathbb{F}_2$ such that $a^2 + a + 1 = 0$. Is there any element $A \in M_2(\mathbb{F}_2)$ such that $A^2 + A + 1 = 0$? ***(3 marks)***

3 Let $f(x) = x^4 + x^3 + x^2 + x + 1$.

(i) By considering the roots of the quadratic $(y-a)(y-b)$, find a factorisation $f(x) = (x^2 + ax + 1)(x^2 + bx + 1)$ in $\mathbb{R}[x]$. ***(3 marks)***

(ii) Show that $f(x)$ has no real roots, hence that the factors found in the previous part are irreducible in $\mathbb{R}[x]$. [Hint: multiply by $(x-1)$.] ***(2 marks)***

(iii) Using unique factorisation in $\mathbb{Z}$, or otherwise, prove that $\sqrt{5} \notin \mathbb{Q}$. ***(2 marks)***

(iv) Prove that $f(x)$ is irreducible in $\mathbb{Q}[x]$. ***(2 marks)***

4 (i) Let R be a non-zero commutative ring. What does it mean for R to be a *field*? ***(1 mark)***

(ii) Give a very brief reason why $\mathbb{Z}/47\mathbb{Z}$ is a field. ***(1 mark)***

(iii) In the field $\mathbb{Z}/47\mathbb{Z}$, find the multiplicative inverse of the element $\overline{11}$. ***(3 marks)***

5 (i) Let F be a field, and $A \in M_{m,n}(F)$, for some integers $m, n \geq 1$. Consider the map $f_A : F^n \to F^m$ defined by $f_A(v) := Av$, for all $v \in F^n$. Prove that f_A is a linear map. ***(2 marks)***

(ii) For A as above, define the null space $\mathrm{Null}(A)$, and prove that it is a subspace of F^n. ***(4 marks)***

(iii) Let V be a vector space over a field F, and $\{v_1, \dots, v_r\}$ a finite set of vectors in V. What is meant by the *span,* $\mathrm{Span}\{v_1, \dots, v_r\}$? ***(1 mark)***

(iv) Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \in M_{2,4}(\mathbb{R})$. Find $v_1, v_2 \in \mathbb{R}^4$ such that $\mathrm{Null}(A) = \mathrm{Span}\{v_1, v_2\}$. ***(3 marks)***

6 Consider the function $m_i : \mathbb{C} \to \mathbb{C}$ defined by $m_i(z) = iz$ for all $z \in \mathbb{C}$. You may assume that m_i is a linear map of $\mathbb{R}$-vector spaces. Write down the matrix representing m_i with respect to the $\mathbb{R}$-basis $\{1, i\}$ of $\mathbb{C}$. ***(1 mark)***

7 Let $V = C^\infty(\mathbb{R})$, the vector space of real-valued functions, with derivatives of all orders, of a real variable. Let $L(V)$ be the ring of linear operators on V, and consider $D \in L(V)$ defined by $D(f) := \dfrac{df}{dx}$.

(i) Is D injective? Is D surjective? Justify your answers. ***(3 marks)***

(ii) What does the First Isomorphism Theorem, applied to the linear map D, tell us? ***(3 marks)***

(iii) Find a basis for the subspace $\ker(D^2 - 3D + 2)$ of V. What is its dimension? ***(2 marks)***

8 The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ has eigenvectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, with eigenvalues $2, 3$ respectively. Calculate the matrix A representing T with respect to the standard basis of $\mathbb{R}^2$. ***(2 marks)***

9 (i) Let $(V, \langle , \rangle)$ be a real inner product space. Let W be a subspace of V. What is meant by the *orthogonal complement* $W^\perp$ of W in V? ***(1 mark)***

(ii) Suppose that $T : V \to V$ is a linear operator such that $\langle T(v), w \rangle = \langle v, T(w) \rangle$ for all $v, w \in V$. Prove that if $T(W) \subseteq W$ then $T(W^\perp) \subseteq W^\perp$. ***(2 marks)***

(iii) Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Prove that if $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear map defined by $T(v) := Av$ for all $v \in \mathbb{R}^2$, and if $\langle , \rangle$ is the standard inner product (dot product), then $\langle T(v), w \rangle = \langle v, T(w) \rangle$ for all $v, w \in \mathbb{R}^2$. Find a basis for $\mathbb{R}^2$ consisting of eigenvectors for T, and confirm directly that they are orthogonal to each other. ***(5 marks)***

End of Question Paper