MAS 220 Sample Exam Solutions

1 (i) $f : G \to H$ is a group homomorphism $\iff f(ab) = f(a)f(b) \ \forall a, b \in G$.

(ii) N is ~~nor~~ normal $\iff gNg^{-1} = N \quad \forall g \in G$

(equivalently $gN = Ng \quad \forall g \in G$).

(iii) $\ker(f) = \{g \in G \mid f(g) = e_H\}$

$f(e_G) = e_H \Rightarrow e_G \in \ker(f)$, so $\ker(f) \neq \emptyset$

Suppose $g_1, g_2 \in \ker(f)$.

Then $f(g_1 g_2) = f(g_1)f(g_2) = e_H e_H = e_H$, so $g_1 g_2 \in \ker(f)$.

Suppose $g \in \ker(f)$.

Then $f(g^{-1}) = f(g)^{-1} = e_H^{-1} = e_H$, so $g^{-1} \in \ker(f)$.

So far, $\ker(f)$ is a subgroup of G.

Now suppose $n \in \ker(f)$ and $g \in G$.

$f(gng^{-1}) = f(g)f(n)f(g^{-1}) = f(g)e_H f(g)^{-1} = f(g)f(g)^{-1} = e_H$,

so $gng^{-1} \in \ker(f)$. This shows that $gNg^{-1} \subseteq N$, so N is normal.

(iv) $\langle (1\,2\,3) \rangle$ is a normal subgroup of S_3.

$\langle (1\,2\,3\,4) \rangle = \{id, (1\,2\,3\,4), (1\,3)(2\,4), (1\,4\,3\,2)\}$ is not normal, because, for instance, it does not contain the conjugate $(1\,3\,2\,4) = (2\,3)(1\,2\,3\,4)(2\,3)^{-1}$ of its element $(1\,2\,3\,4)$.

2 (i) $(ab)^2 = (ab)(ab) = ((ab)a)b = (a(ba))b = (a(ab))b$ (since R is commutative)

$= ((aa)b)b = (aa)(bb) = a^2b^2$.

(ii) $(a+b)^2 = (a+b)(a+b) = a(a+b) + b(a+b) = a^2 + ab + ba + b^2$

$= a^2 + ab + ab + b^2$ since R is commutative

$= a^2 + (1+1)(ab) + b^2 = a^2 + 0(ab) + b^2 = a^2 + 0 + b^2 = a^2 + b^2$.

(iii) $|R| = 2^4 = 16$, since there are 4 entries, each of which can be chosen in 2 ways.

By (i) and (ii), we can't succeed unless $AB \neq BA$.

Try $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, for which $AB = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

(swapping the rows of B – note that A is an elementary matrix)

while $BA = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ (swapping the columns of B).

Check: $(AB)^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, while $A^2B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

so $(AB)^2 \neq A^2B^2$.

$$(A+B)^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

while $A^2+B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $(A+B)^2 \neq A^2+B^2$.

(iv) In $\mathbb{F}_2$, $0^2+0^2+1=1\neq 0$, and $1^2+1^2+1=1\neq 0$.

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. $A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$$A^2+A+I = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

3 (i) $(x^2+ax+1)(x^2+bx+1) = x^4+(a+b)x^3+(2+ab)x^2+(a+b)x+1$

so $a+b=1$ and $ab=-1$.

$(y-a)(y-b) = y^2-(a+b)y+ab = y^2-y-1$

Roots $\frac{1}{2}(1\pm\sqrt{5})$, so $a,b = \frac{1}{2}(1\pm\sqrt{5})$

$f(x) = (x^2+\frac{1}{2}(1+\sqrt{5})x+1)(x^2+\frac{1}{2}(1-\sqrt{5})x+1)$ in $\mathbb{R}[x]$.

(ii) $f(x)(x-1) = x^5-1$, so any real root of $f(x)$ is a real root of x^5-1, which could only be 1, but $f(1)=5\neq 0$, so $f(x)$ has no real roots, hence no linear factors, so the quadratic factors found in (i) must be irreducible.

(iii) If $\sqrt{5} = \frac{a}{b} \in \mathbb{Q}$ then $5 = \frac{a^2}{b^2}$ so $a^2 = 5b^2$

On left, ~~prime~~ irreducible factor 5 occurs an even number of times. On right, it occurs an odd number of times, contradicting unique factorisation.

(iv) Any linear factor of $f(x)$ in $\mathbb{Q}[x]$ would be in $\mathbb{R}[x]$, contrary to (ii), so if $f(x)$ is reducible in $\mathbb{Q}[x]$, it must be a product of two irreducible quadratic factors, which we may assume to be monic (leading coefficient 1), say $f(x)=g(x)h(x)$.

By (i), (ii) and uniqueness of factorisation into irreducibles in $\mathbb{R}[x]$, must have $g(x), h(x) = (x^2+\frac{1}{2}(1\pm\sqrt{5})x+1)$.

This is impossible, since if these were in $\mathbb{Q}[x]$ then $\gamma = \frac{1}{2}(1+\sqrt{5}) \in \mathbb{Q}$, so $\underset{2\gamma-1}{\sqrt{5}} \in \mathbb{Q}$, contrary to (iii).

[By the way, γ is the "golden ratio".]

4 (i) R is a field $\Leftrightarrow$ every non-zero $a \in R$ has a multiplicative inverse in R, i.e. an element $b \in R$ such that $ab = 1$.

(ii) $\mathbb{Z}/47\mathbb{Z}$ is a field because 47 is a prime number
not required (an irreducible element of the Euclidean domain $\mathbb{Z}$).

(iii) $47 = 4(11) + 3$
$11 = 4(3) - 1$.
$1 = 4(3) - 11 = 4(47 - 4(11)) - 11 = 4(47) - 17(11)$
$(-17)(11) \equiv 1 \pmod{47}$
so $\overline{11}^{-1} = \overline{-17} = \overline{30}$.

5 (i) $f_A(v_1 + v_2) = A(v_1 + v_2) = Av_1 + Av_2 = f_A(v_1) + f_A(v_2) \quad \forall v_1, v_2 \in F^n$.
$f_A(\lambda v) = A(\lambda v) = \lambda(Av) = \lambda f_A(v) \quad \forall v \in F^n, \lambda \in F$.
Hence f_A is linear.

(ii) $\mathrm{Null}(A) = \{\underline{x} \in F^n \mid A\underline{x} = \underline{0}_m\}$
$A\underline{0}_n = \underline{0}_m$, so $\underline{0}_n \in \mathrm{Null}(A)$, so $\mathrm{Null}(A) \neq \emptyset$.
If $\underline{x}_1, \underline{x}_2 \in \mathrm{Null}(A)$ then $A(\underline{x}_1 + \underline{x}_2) = A\underline{x}_1 + A\underline{x}_2 = \underline{0} + \underline{0} = \underline{0}$,
so $\underline{x}_1 + \underline{x}_2 \in \mathrm{Null}(A)$.
If $\underline{x} \in \mathrm{Null}(A)$ and $\lambda \in F$ then $A(\lambda\underline{x}) = \lambda A\underline{x} = \lambda\underline{0} = \underline{0}$,
so $\lambda\underline{x} \in \mathrm{Null}(A)$. Hence $\mathrm{Null}(A)$ is a subspace of F^n.

(iii) $\mathrm{Span}\{v_1, \dots, v_r\} = \{\sum_{i=1}^{r} \lambda_i v_i \mid \lambda_1, \dots, \lambda_r \in F\}$

(iv) $$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\mathrm{Null}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

$$\cancel{8 \quad A\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}}$$

$$\cancel{\Rightarrow A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 6 \\ 4 & 9 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 6 & -1 \end{bmatrix}}.$$

6 $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

7 (i) D is not injective, since it has a non-zero kernel consisting of constant functions.
D is surjective, since given any $g \in V$, $g = D(f)$ where $f(x) := \int_0^x g(t)\,dt$.

(ii) F.I.T. tells us $\underbrace{\dfrac{C^\infty(\mathbb{R})}{\{\text{const}\}}}_{\ker D} \cong \underbrace{C^\infty(\mathbb{R})}_{\operatorname{Im} D}$

$$f + C \longmapsto \frac{df}{dx}$$

(iii) $f \in \ker(D^2 - 3D + 2) \iff \frac{d^2f}{dx^2} - 3\frac{df}{dx} + 2f = 0.$
Auxiliary polynomial $\lambda^2 - 3\lambda + 2 = (\lambda-2)(\lambda-1)$ Roots $\lambda = 1, 2$.
General solution $Ae^x + Be^{2x}$
$\ker(D^2 - 3D + 2)$ has basis $\{e^x, e^{2x}\}$ and is 2-dimensional.

8 $A\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 6 \\ 4 & 9 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 6 & -1 \end{bmatrix}.$

9 (i) $W^{\perp} = \{ v \in V \mid \langle w, v \rangle = 0 \ \forall w \in W \}$

(ii) Suppose $u \in W^{\perp}$. Must show $T(u) \in W^{\perp}$.

But for any $w \in W$, $\langle w, T(u) \rangle = \langle T(w), u \rangle = 0$ since $T(W) \subseteq W \Rightarrow T(w) \in W$, and since $u \in W^{\perp}$.

Hence $T(u) \in W^{\perp}$, as required.

(iii) $\langle T(v), w \rangle = \langle Av, w \rangle = (Av)^T w = (v^T A^T) w = v^T (A^T w)$

$= v^T (Aw)$ since A is symmetric

$= \langle v, Aw \rangle = \langle v, T(w) \rangle$.

$$\det(xI - A) = \begin{vmatrix} (x-1) & -2 \\ -2 & (x-1) \end{vmatrix} = (x-1)^2 - 4 = x^2 - 2x - 3 = (x-3)(x+1)$$

Eigenvalues $\lambda = 3, -1$.

$\underline{\lambda = 3}$ $A - \lambda I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$. Eigenspace Span$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$\underline{\lambda = -1}$ $A - \lambda I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$. Eigenspace Span$\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$.

Basis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ consisting of eigenvectors for T.

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1(1) + 1(-1) = 0$, so they are orthogonal.