

MAS 220 Sample Exam Solutions

1 (i) $f: G \rightarrow H$ is a group homomorphism $\Leftrightarrow f(ab) = f(a)f(b) \quad \forall a, b \in G$.

(ii) N is ~~normal~~ normal $\Leftrightarrow gNg^{-1} = N \quad \forall g \in G$
(equivalently $gN = Ng \quad \forall g \in G$).

(iii) $\ker(f) = \{g \in G \mid f(g) = e_H\}$
 $f(e_G) = e_H \Rightarrow e_G \in \ker(f)$, so $\ker(f) \neq \emptyset$
Suppose $g_1, g_2 \in \ker(f)$.

Then $f(g_1 g_2) = f(g_1) f(g_2) = e_H e_H = e_H$, so $g_1 g_2 \in \ker(f)$.
Suppose $g \in \ker(f)$.

Then $f(g^{-1}) = f(g)^{-1} = e_H^{-1} = e_H$, so $g^{-1} \in \ker(f)$.

So far, $\ker(f)$ is a subgroup of G .

Now suppose $n \in \ker(f)$ and $g \in G$.

$$f(gng^{-1}) = f(g) f(n) f(g^{-1}) = f(g) e_H f(g^{-1}) = f(g) f(g)^{-1} = e_H,$$

so $gng^{-1} \in \ker(f)$. This shows that $gNg^{-1} \subseteq N$, so N is normal.

(iv) $\langle (123) \rangle$ is a normal subgroup of S_3 .

$\langle (1234) \rangle = \{id, (1234), (13)(24), (1432)\}$ is not normal, because, for instance, it does not contain the conjugate

$$(1324) = (23)(1234)(23)^{-1} \text{ of its element } (1234).$$

2 (i) $(ab)^2 = (ab)(ab) = ((ab)a)b = (a(ba))b = (a(ab))b$ (since R is commutative)
 $= (aa)b = (a^2)b = a^2 b^2$.

(ii) $(a+b)^2 = (a+b)(a+b) = a(a+b) + b(a+b) = a^2 + ab + ba + b^2$
 $= a^2 + ab + ab + b^2$ since R is commutative
 $= a^2 + (1+1)ab + b^2 = a^2 + 0(ab) + b^2 = a^2 + 0 + b^2 = a^2 + b^2$.

(iii) $|R| = 2^4 = 16$, since there are 4 entries, each of which can be chosen in 2 ways.

By (i) and (ii), we can't succeed unless $AB \neq BA$.

Try $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, for which $AB = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

(swapping the rows of B - note that A is an elementary matrix)

while $BA = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ (swapping the columns of B).

Check: $(AB)^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, while $A^2 B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,
so $(AB)^2 \neq A^2 B^2$.

$$(A+B)^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

while $A^2 + B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $(A+B)^2 \neq A^2 + B^2$.

(iv) In \mathbb{F}_2 , $0^2 + 0^2 + 1 = 1 \neq 0$, and $1^2 + 1^2 + 1 = 1 \neq 0$.

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. $A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$$A^2 + A + I = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

3 (i) $(x^2 + ax + 1)(x^2 + bx + 1) = x^4 + (a+b)x^3 + (2+ab)x^2 + (a+b)x + 1$
so $a+b=1$ and $ab=-1$.

$$(y-a)(y-b) = y^2 - (a+b)y + ab = y^2 - y - 1$$

Roots $\frac{1}{2}(1 \pm \sqrt{5})$, so $a, b = \frac{1}{2}(1 \pm \sqrt{5})$

$$f(x) = (x^2 + \frac{1}{2}(1+\sqrt{5})x + 1)(x^2 + \frac{1}{2}(1-\sqrt{5})x + 1) \text{ in } \mathbb{R}[x].$$

(ii) $f(x)(x-1) = x^5 - 1$, so any real root of $f(x)$ is a real root of $x^5 - 1$, which could only be 1, but $f(1) = 5 \neq 0$, so $f(x)$ has no real roots, hence no linear factors, so the quadratic factors found in (i) must be irreducible.

(iii) if $\sqrt{5} = \frac{a}{b} \in \mathbb{Q}$ then $5 = \frac{a^2}{b^2}$ so $a^2 = 5b^2$

On left, ~~prime~~ ^{irreducible} factor 5 occurs an even number of times.

On right, it occurs an odd number of times, contradicting unique factorisation.

(iv) Any linear factor of $f(x)$ in $\mathbb{Q}[x]$ would be in $\mathbb{R}[x]$, contrary to (ii), so if $f(x)$ is reducible in $\mathbb{Q}[x]$, it must be a product of two irreducible quadratic factors, which we may assume to be monic (leading coefficient 1), say $f(x) = g(x)h(x)$.

By (i), (ii) and uniqueness of factorisation into irreducibles in $\mathbb{R}[x]$, must have $g(x), h(x) = (x^2 + \frac{1}{2}(1 \pm \sqrt{5})x + 1)$.

This is impossible, since if these were in $\mathbb{Q}[x]$ then $\gamma = \frac{1}{2}(1 + \sqrt{5}) \in \mathbb{Q}$, so $\sqrt{5} \in \mathbb{Q}$, contrary to (iii).

$2\gamma - 1$

[By the way, γ is the "golden ratio".]

4 (i) R is a field \Leftrightarrow every non-zero $a \in R$ has a multiplicative inverse in R , i.e. an element $b \in R$ such that $ab = 1$.

(ii) $\mathbb{Z}/47\mathbb{Z}$ is a field because 47 is a prime number
not required \rightarrow (an irreducible element of the Euclidean domain \mathbb{Z}).

(iii) $47 = 4(11) + 3$

$11 = 4(3) - 1$

$1 = 4(3) - 11 = 4(47 - 4(11)) - 11 = 4(47) - 17(11)$

$(-17)(11) \equiv 1 \pmod{47}$

so $\overline{11}^{-1} = \overline{-17} = \overline{30}$.

5 (i) $f_A(v_1+v_2) = A(v_1+v_2) = Av_1 + Av_2 = f_A(v_1) + f_A(v_2) \quad \forall v_1, v_2 \in F^n$

$f_A(\lambda v) = A(\lambda v) = \lambda(Av) = \lambda f_A(v) \quad \forall v \in F^n, \lambda \in F$

Hence f_A is linear.

(ii) $\text{Null}(A) = \{ \underline{x} \in F^n \mid A\underline{x} = \underline{0}_m \}$

$A\underline{0}_n = \underline{0}_m$, so $\underline{0}_n \in \text{Null}(A)$, so $\text{Null}(A) \neq \emptyset$.

If $\underline{x}_1, \underline{x}_2 \in \text{Null}(A)$ then $A(\underline{x}_1 + \underline{x}_2) = A\underline{x}_1 + A\underline{x}_2 = \underline{0} + \underline{0} = \underline{0}$,
 so $\underline{x}_1 + \underline{x}_2 \in \text{Null}(A)$.

If $\underline{x} \in \text{Null}(A)$ and $\lambda \in F$ then $A(\lambda \underline{x}) = \lambda A\underline{x} = \lambda \underline{0} = \underline{0}$,
 so $\lambda \underline{x} \in \text{Null}(A)$. Hence $\text{Null}(A)$ is a subspace of F^n .

(iii) $\text{Span}\{v_1, \dots, v_r\} = \left\{ \sum_{i=1}^r \lambda_i v_i \mid \lambda_1, \dots, \lambda_r \in F \right\}$

(iv) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

$\text{Null}(A) = \text{Span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

~~$A \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$~~
 ~~$\Rightarrow A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 6 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 6 & -1 \end{bmatrix}$~~

$$\underline{\underline{6}} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

7 (i) D is not injective, since it has a non-zero kernel consisting of constant functions.

D is surjective, since given any $g \in V$, $g = D(f)$
 where $f(x) := \int_0^x g(t) dt$.

(ii) F.I.T. tells us $\frac{C^\infty(\mathbb{R})}{\ker D} \cong \text{Im } D$

~~\mathbb{R}~~
 $f + C \mapsto \frac{df}{dx}$

(iii) $f \in \ker(D^2 - 3D + 2) \Leftrightarrow \frac{d^2f}{dx^2} - 3\frac{df}{dx} + 2f = 0$.

Auxiliary polynomial $\lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$ Roots $\lambda = 1, 2$.

General solution $Ae^x + Be^{2x}$

$\ker(D^2 - 3D + 2)$ has basis $\{e^x, e^{2x}\}$ and is 2-dimensional.

8 $A \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 6 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 6 & -1 \end{bmatrix}$.

9 (i) $W^\perp = \{v \in V \mid \langle w, v \rangle = 0 \ \forall w \in W\}$

(ii) Suppose $u \in W^\perp$. Must show $T(u) \in W^\perp$.

But for any $w \in W$, $\langle w, T(u) \rangle = \langle T(w), u \rangle = 0$
since $T(W) \subseteq W \Rightarrow T(w) \in W$, and since $u \in W^\perp$.

Hence $T(u) \in W^\perp$, as required.

(iii) $\langle T(v), w \rangle = \langle Av, w \rangle = (Av)^T w = (v^T A^T) w = v^T (A^T w)$
 $= v^T (Aw)$ since A is symmetric
 $= \langle v, Aw \rangle = \langle v, T(w) \rangle$.

$$\det(xI - A) = \begin{vmatrix} (x-1) & -2 \\ -2 & (x-1) \end{vmatrix} = (x-1)^2 - 4 = x^2 - 2x - 3 = (x-3)(x+1)$$

Eigenvalues $\lambda = 3, -1$.

$\lambda = 3$ $A - \lambda I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$. Eigenspace $\text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$\lambda = -1$ $A - \lambda I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$. Eigenspace $\text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

Basis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ consisting of eigenvectors for T .

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1(1) + 1(-1) = 0, \text{ so they are orthogonal.}$$