LIFTING PUZZLES AND CONGRUENCES OF IKEDA AND IKEDA-MIYAWAKI LIFTS

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Abstract. We show how many of the congruences between Ikeda lifts and non-Ikeda lifts, proved by Katsurada, can be reduced to congruences involving only forms of genus 1 and 2, using various liftings predicted by Arthur’s multiplicity conjecture. Similarly, we show that conjectured congruences between Ikeda-Miyawaki lifts and non-lifts can often be reduced to congruences involving only forms of genus 1, 2 and 3.

1. Introduction

For $k, g \geq 2$ even, let $f \in S_{2k-g}(\text{SL}(2, \mathbb{Z}))$ be a normalised Hecke eigenform. Duke and Imamoglu conjectured the existence of a cuspidal Hecke eigenform $F \in S_k(\text{Sp}_g(\mathbb{Z}))$ (a Siegel modular form of genus $g$) such that its standard $L$-function

$$L(s, F, \text{St}) = \zeta(s) \prod_{i=1}^{g} L(f, s + (k - i)).$$

The existence of this $F$ was proved by Ikeda [Ik1], who gave its Fourier expansion, and we call it the Ikeda lift. In the case $g = 2$ it was already known, as the Saito-Kurokawa lift. Katsurada [Ka1] proved that if $k \geq 2g + 4$ and $q > 2k$ is a prime number such that, for some divisor $q | g$ in a sufficiently large number field,

$$\text{ord}_q(L_{\text{alg}}(f, k)) \prod_{i=1}^{(g/2) - 1} L_{\text{alg}}(2i + 1, f, \text{St}) > 0,$$

then, under certain weak conditions, there is a congruence mod $q$ of Hecke eigenvalues, between $F$ and some Hecke eigenform, in the same space $S_k(\text{Sp}_g(\mathbb{Z}))$, that is not an Ikeda lift. Here the $L$-values have been normalised by dividing them by particular choices of Deligne periods. This generalises his earlier work on congruences for Saito-Kurokawa lifts (for which only the factor $L(f, k)$ appears), and similarly it uses a pullback formula for an Eisenstein series of genus $2g$ to which a certain differential operator has been applied. The $L$-values arise as factors in a formula for the Petersson norm of $F$, which had been proved by Kohnen and Skoruppa for Saito-Kurokawa lifts, and for $g > 2$ was conjectured by Ikeda and proved by Katsurada and Kawamura. For $g = 2$, congruences were proved independently by Brown [Br], who used them to construct elements in Selmer groups supporting the Bloch-Kato conjecture applied to the critical value $L(f, k)$, which for $g = 2$ is immediately to the right of the central point.

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As \( g \) increases, the value \( s = k \) migrates further and further to the right in the critical range \( 1 \leq s \leq 2k - g \). (Of course, we must adjust \( k \) if we want to keep the weight \( 2k - g \) the same to look at a fixed \( f \).) Prime divisors of the algebraic parts of these critical values appear as the moduli of congruences conjectured by Harder [H, vdG], which support the Bloch-Kato conjecture for these critical values. These congruences of Hecke eigenvalues involve vector-valued Siegel modular forms of genus 2, and may be viewed as being congruences of Hecke eigenvalues between cuspidal automorphic representations of \( \mathrm{GSp}_2(A) \) and representations induced from the Levi subgroup \( \mathrm{GL}_1 \times \mathrm{GL}_2 \) of the Siegel parabolic subgroup [BD, §7]. The Hecke eigenvalues of these induced representations involve those of \( f \). Faber and van der Geer [FvdG] computed many Hecke eigenvalues of vector-valued Siegel modular forms of genus 2, providing numerical evidence for many instances of Harder’s conjecture. The original example, with \( 41 \mid L_{\text{alg}}(f, 14), \) for \( f \) of weight 22, has been proved by Chenevier and Lannes [CL].

Prime divisors of \( L_{\text{alg}}(2i + 1, f, \text{St}) \) also appear as moduli of conjectural congruences of Hecke eigenvalues involving only genus 2 forms, in general vector-valued, in fact this applies to \( L_{\text{alg}}(r, f, \text{St}) \) for all odd \( r \) from 3 to \( 2k - g - 1 \). The congruences are between cusp forms and Klingen-Eisenstein series, and again may be viewed as being between cuspidal and induced automorphic representations of \( \mathrm{GSp}_2(A) \), this time for the Klingen parabolic subgroup [BD, §6]. The first example, for \( q = 71 \) and \( f \) of weight 20, was proved by Kurokawa [Ku], and Mizumoto proved a more general result [Miz]. Their work involved scalar-valued forms of genus 2, and the rightmost critical value of \( L(s, f, \text{St}) \). One deals with critical values further to the left by increasing the “vector part” \( j \) of the weight. Satoh proved a congruence mod 343 in a \( j = 2 \) case [Sa], and further instances, for other \( j \), were proved in [Du].

Poor, Ryan and Yuen [PRY] computed the Euler factors at 2 of the standard \( L \)-functions of the seven cuspidal Hecke eigenforms in \( S_{16}(\mathrm{Sp}_4(\mathbb{Z})) \) (genus 4). Two of these forms are Ikeda lifts, while another two are lifts of pairs of genus 1 forms, of a type conjectured by Miyawaki and proved by Ikeda. The remaining three were more mysterious, but the Euler 2-factors of their standard \( L \)-functions factored in such a way as to suggest that they were lifts of some previously unknown kind. A. Mellit suggested to T. Ibukiyama that one of them should be lifted from a vector-valued Siegel modular form of genus 2, whose spinor \( L \)-function would appear in the standard \( L \)-function of the lift. Ibukiyama [Ib] then made two conjectures on scalar-valued genus 4 lifts of genus 2 vector-valued forms, in whose standard \( L \)-functions the spinor and standard \( L \)-functions of the lifted form, respectively, would appear. For the “standard” lift, a genus 1 form is also involved. He checked that these conjectures produce precisely the Euler 2-factors computed by Poor, Ryan and Yuen, and generalised the conjectures to predict scalar-valued lifts, to higher genus, of genus 1 and (vector-valued) genus 2 forms.

Reconsidering Katsurada’s congruences between Ikeda lifts and non-Ikeda lifts, the occurrence of the same \( L \)-values in conjectural congruences involving only genus 1 and genus 2 forms, and the apparent existence of scalar-valued, higher genus lifts of such forms, suggest the question of whether these things are related. Could the non-Ikeda lifts in Katsurada’s congruences actually be lifts of the type proposed by Ibukiyama? For \( L(f, k) \), Ibukiyama’s “standard lift” indeed explains Katsurada’s congruence as a “lift” of Harder’s. If \( 4 \mid g \) then for \( L((g/2) + 1, f, \text{St}) \) (the factor
for \( i = \frac{q}{2} \), Ibukiyama’s “spinor lift” likewise explains Katsurada’s congruence as a lift of a congruence of Kurokawa-Mizumoto type. In fact, generalising the spinor lift to lift the genus 1 form as well as a genus 2 form, we may similarly account for congruences involving \( L(2i + 1, f, St) \), for \( \frac{q}{2} \leq i \leq \frac{g}{2} - 1 \), i.e. for about half the values of \( i \).

We consider also congruences between Ikeda-Miyawaki lifts and non-Ikeda-Miyawaki lifts, conjectured by Ibukiyama, Katsurada, Poor and Yuen [IKPY]. They could be proved in the same manner as those between Ikeda lifts and non-Ikeda-lifts, if one knew a conjecture of Ikeda on the Petersson norm of an Ikeda-Miyawaki lift. The moduli are large prime divisors of \( L_{\text{alg}}(f \otimes \text{Sym}^2 h, 2k + 2n) \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} L_{\text{alg}}(2i + 1, f, St) \), where \( f \) and \( h \) are genus 1 forms of weights \( 2k \) and \( k + n + 1 \) respectively, and the Ikeda-Miyawaki lift is of genus \( 2n + 1 \), weight \( k + n + 1 \). Again, it appears that in many cases the non-Ikeda-Miyawaki lift should in fact be some other kind of lift. For \( L_{\text{alg}}(f \otimes \text{Sym}^2 h, 2k + 2n) \) we “lift” a genus 3 generalisation of Harder’s conjecture, worked out by Harder himself in collaboration with the authors of [BFvdG], in which it is Conjecture 10.8. Their computations of genus 3 Hecke eigenvalues, together with \( L \)-value approximations by Mellit (subsequently confirmed by exact computations in [IKPY]), provided numerical support for their conjecture in seventeen cases. For \( L_{\text{alg}}(2i + 1, f, St) \), with \( \lfloor \frac{n}{2} \rfloor \leq i \leq n - 1 \), we again lift congruences of Kurokawa-Mizumoto type.

We may now appear to have a proliferation of unsupported conjectures on the existence of various lifts. But we show how they all fit into Arthur’s endoscopic classification of the discrete spectrum of \( \text{Sp}_g(\mathbb{Q}) \backslash \text{Sp}_g(A) \), and would be consequences of his conjectural multiplicity formula. Actually, for certain groups including \( \text{Sp}_g \), Arthur has proved a version of his multiplicity formula [A, Theorem 1.5.2]. But its equivalence to the version applied here is dependent on an as-yet unproved equivalence between two ways of defining and parametrising an \( L \)-packet at \( \infty \), as explained following [CR, Conjecture 3.23].

After preliminaries on Arthur’s endoscopic classification and multiplicity formula, in Sections 3 and 4, we apply them in Section 5 to obtain all the various lifts (including those of Ikeda and Ikeda-Miyawaki), conditional on the as yet unproved multiplicity formula. The compatibility of the Ikeda lift with Arthur’s conjecture was already mentioned in [Ik1, §14], and Ibukiyama looked at the Arthur parameters of his proposed lifts in [Ib, §3.4], without checking the multiplicity formula. In Section 6 we look at the congruences between Ikeda lifts and non-Ikeda lifts proved by Katsurada, and those between Ikeda-Miyawaki lifts and non-Ikeda-Miyawaki lifts conjectured in [IKPY]. Finally, in Section 7 we describe in more detail how some of these congruences can be accounted for in the manner indicated above.

The Hecke algebra for Siegel modular forms of genus \( g \) is generated by Hecke operators for each prime \( p \), traditionally denoted \( T(p) \) and \( T_i(p^s) \) for \( 1 \leq i \leq g \). Strictly speaking, our approach only accounts for congruences between Hecke eigenvalues for the \( T_i(p^s) \), not the \( T(p) \). This is because we produce Arthur parameters for \( G = \text{Sp}_g \) (with \( \hat{G} = \text{SO}(g + 1, g) \)) rather than for \( G = \text{GSp}_g \) (with \( \hat{G} = \text{Spin}(g + 1, g) \)). The Siegel modular forms we consider are all eigenforms for the \( T(p) \) as well as the \( T_i(p^s) \), but we cannot deduce from this the congruence of the \( T(p) \) Hecke eigenvalues.
2. SYMPLECTIC AND SPECIAL ORTHOGONAL GROUPS

Let $G = \text{Sp}_g = \{ h \in M_{2g} : {}^tJhJ = J \}$, where

$$J_{i,2g+1-i} = \begin{cases} 1 & \text{if } 1 \leq i \leq g; \\ -1 & \text{if } g + 1 \leq i \leq 2g, \end{cases}$$

and all other entries are 0. It has a maximal torus $T$ comprising elements of the form diag$(t_1, \ldots, t_g, t_g^{-1}, \ldots, t_1^{-1})$, which is mapped to $t_i$ by characters $e_i$, for $1 \leq i \leq g$, which span the character group $X^*(T)$. The cocharacter group $X_*(T)$ is spanned by $\{ f_1, \ldots, f_g \}$, where $f_i : t \mapsto \text{diag}(t, 1, \ldots, 1, t^{-1})$, etc. so $\langle e_i, f_j \rangle = \delta_{ij}$. We can order the roots so that the positive roots are $\Phi^+_G = \{ e_i - e_j : i \leq j \} \cup \{ 2e_i : 1 \leq i \leq g \} \cup \{ e_i + e_j : i < j \}$, and the simple roots $\Delta_G = \{ e_1 - e_2, e_2 - e_3, \ldots, e_{g-1} - e_g, 2e_g \}$. The simple coroots (in order) are $\{ \tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_{g-1} - \tilde{e}_g, 2\tilde{e}_g \}$. Note that for any root $\beta$ with coroot $\hat{\beta}$, we have $\langle \beta, \hat{\beta} \rangle = 2$.

We see then that the root systems of $G$ and $\hat{G}$ are dual to each other, so $\hat{G}$ is, as the notation indicates, the Langlands dual of $G$. The isomorphisms $X^*(T) \simeq X_*(T)$ and $X^*(\hat{T}) \simeq X_*(\hat{T})$ are such that $\tilde{e}_i \leftrightarrow f_i$ and $e_i \leftrightarrow \tilde{f}_i$, respectively.

Let $\mathfrak{h}_g$ be the Siegel upper half space of $g$ by $g$ complex symmetric matrices with positive-definite imaginary part. For $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{Sp}_g(\mathbb{Z})$ and $Z \in \mathfrak{h}_g$, let $M(Z) := (AZ + B)(CZ + D)^{-1}$ and $J(M, Z) := CZ + D$. Let $V$ be the space of a representation $\rho$ of $\text{GL}(g, \mathbb{C})$. A holomorphic function $f : \mathfrak{h}_g \to V$ is said to belong to the space $M_\rho(\text{Sp}_g(\mathbb{Z}))$ of Siegel modular forms of genus $g$ and weight $\rho$ if

$$f(M(Z)) = \rho(J(M, Z))f(Z) \quad \forall M \in \text{Sp}_g(\mathbb{Z}), Z \in \mathfrak{h}_g,$$

and, in the case $g = 1$, if it is holomorphic at the cusps. If $g > 1$, the Siegel operator $\Phi$ on $M_\rho(\text{Sp}_g(\mathbb{Z}))$ is defined by

$$\Phi f(z) = \lim_{t \to \infty} f \left( \begin{bmatrix} z & 0 \\ 0 & it \end{bmatrix} \right) \quad \text{for } z \in \mathfrak{h}_{g-1}, t \in \mathbb{R}.$$

The kernel of $\Phi$, denoted $S_\rho(\text{Sp}_g(\mathbb{Z}))$, is the space of Siegel cusp forms of genus $g$ and weight $\rho$. When $\rho = \text{det}^k$, the forms are scalar valued, of weight $k$, and $S_\rho(\text{Sp}_g(\mathbb{Z}))$ is denoted $S_k(\text{Sp}_g(\mathbb{Z}))$. 
3. Arthur’s endoscopic classification

Let $G = \text{Sp}(g)$ as above, so $\hat{G} = \text{SO}(g + 1, g)$. Let $\text{St} : \hat{G} \to \text{SL}(2g + 1)$ be the standard inclusion homomorphism. Let $X(\hat{G})$ be the set of $(c_{\pi})$, indexed by places $v$ of $\mathbb{Q}$, such that for finite $p$, $c_{\pi}$ is a semisimple conjugacy class in $\hat{G}(\mathbb{C})$, and $c_{\infty}$ is a semisimple conjugacy class in $\text{Lie}(\hat{G}(\mathbb{C}))$. Let $\Pi(G)$ be the set of irreducible representations $\pi$ of $G(\mathbb{A})$ such that $\pi_{\infty}$ is unitary and each $\pi_p$, for finite primes $p$, is smooth and unramified, i.e. has a non-zero $G(\mathbb{Z}_p)$-fixed vector. Let $\Pi_{\text{disc}}(G)$ be the subset of those occurring discretely in $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$. Given $\pi \in \Pi_{\text{disc}}(G)$, let $c(\pi) = (c_{\pi}(v))$, where for finite $p$, $c_{\pi}(v)$ is the Satake parameter of $\pi_p$, and $c_{\infty}(\pi)$ is the infinitesimal character of $\pi_{\infty}$. We may do something similar with $G$ replaced by $\text{PGL}(m)$ and $\hat{G}$ by $\text{PGL}(m) = \text{SL}(m)$, or with $G$ replaced by $\text{SO}(g + 1, g)$ and $\hat{G}$ by $\text{Sp}(g)$. Let $\text{St} : \text{Sp}(g) \to \text{SL}(2g)$, or with $G$ and $\hat{G}$ both replaced by $\text{SO}(g, g)$, $\text{St} : \text{SO}(g, g) \to \text{SL}(2g)$.

As an example, if $\pi_f$ is the cuspidal automorphic representation of $\text{PGL}(2)(\mathbb{A})$ associated with a normalised, cuspidal Hecke eigenform $f$ for $\text{SL}(2, \mathbb{R})$, then $c_{\pi}(\pi_f) = \text{diag}(\alpha_p, \alpha_p^{-1})$ of weight $k$ for $\text{SL}(2, \mathbb{Z})$, then $c_{\pi}(\pi_f) = \text{diag}(\alpha_p, \alpha_p^{-1})$, where $\alpha_p = p^{k-1/2}(\alpha_p + \alpha_p^{-1})$, and $c_{\infty}(\pi_f) = \text{diag}((k - 1)/2, -(k - 1)/2)$. We have $L(f, s + 1/2) = \prod_p \text{det}(I - c_{\pi_f}(\pi_f))\gamma_p^{-1}$. In this example we may also think of $\text{PGL}(2)$ as $\text{SO}(2, 1)$, and $\text{SL}(2)$ as $\text{SO}(2, 1) = \text{Sp}_1$. If instead we consider the cuspidal automorphic representation $\pi_f^\alpha$ of $\text{Sp}_1(\mathbb{A}) = \text{SL}_2(\mathbb{A})$ associated with $f$ then $c_{\pi_f}(\pi_f^\alpha) = \text{diag}(\alpha_p, \alpha_p^{-1})$, $\alpha_p = p^{k-1/2}(\alpha_p + \alpha_p^{-1})$, and $\prod_p \text{det}(I - \text{St}(c_{\pi_f}(\pi_f^\alpha)))\gamma_p^{-1}$ is the standard $L$-function $L(s, f, \text{St}) = L(s + 1/2, \text{Sym}^2 f)$, while $c_{\infty}(\pi_f^\alpha) = \text{diag}(k - 1, 0, 1 - k)$, which can be thought of as $(k - 1)e_1$.

By Arthur’s symplectic-orthogonal alternative [CR, Theorem 3.9], given any $\pi \in \Pi_{\text{cusp}}(\text{PGL}(m))$ (the subset of cuspidal representations in $\Pi_{\text{disc}}(\text{PGL}(m))$), there is a $G^\pi = \begin{cases} \text{Sp}(m-1)/2 & \text{if } m \text{ is odd;} \\ \text{SO}(m/2, m/2) \text{ or } \text{SO}((m/2) + 1, m/2) & \text{if } m \text{ is even,} \end{cases}$ and $\pi' \in \pi_{\text{disc}}(G^\pi)$ such that $c(\pi) = \text{St}(c(\pi'))$.

Following [CR, §3.11] (where more generally $G$ is a classical semisimple group over $\mathbb{Z}$), let $\Psi_{\text{glob}}(G)$ be the set of quadruples $(k_i, (n_i), (d_i), (\pi_i))$, where $1 \leq k_i \leq 2g + 1$, $k_i$ an integer, $n_i \geq 1$ are integers with $\sum_{i=1}^{k_i} n_i = 2g + 1$, $d_i | n_i$ and each $\pi_i \in \Pi_{\text{cusp}}(\text{PGL}(n_i/d_i))$ is a self-dual, cuspidal, automorphic representation of $\text{PGL}(n_i/d_i)(\mathbb{A})$. There are two conditions:

(1) if $(n_i, d_i) = (n_j, d_j)$ with $i \neq j$, then $\pi_i \neq \pi_j$;

(2) $d_i$ is odd if $G^{\pi_i}$ is orthogonal, while $d_i$ is even if $G^{\pi_i}$ is symplectic.

An element $\psi \in \Psi_{\text{glob}}(G)$ is called a global Arthur parameter. We write

$\psi = \pi_1[d_1] \oplus \pi_2[d_2] \oplus \ldots \oplus \pi_k[d_k],$

where there is an equivalence relation, such that for the equivalence class of $\psi$ the order of the summands is unimportant. If $\pi_i$ is the trivial representation we just write $[d_i]$ for $\pi_i[d_i]$, and we just write $\pi$ for $\pi_i[1]$.
To a global Arthur parameter $\psi \in \Psi_{\text{glob}}(G)$, we associate a homomorphism

$$\rho_\psi : \prod_{i=1}^{k} (\text{SL}(n_i/d_i) \times \text{SL}(2)) \to \text{SL}_{2g+1},$$

well-defined up to conjugation in $\text{SL}_{2g+1}(\mathbb{C})$, namely $\bigoplus_{i=1}^{k} (\mathbb{C}^{n_i/d_i} \otimes \text{Sym}^{d_i-1}(\mathbb{C}^2))$. Hence we get a map

$$\rho_\psi : \prod_{i=1}^{k} (\mathcal{X}(\text{SL}(n_i/d_i)) \times \mathcal{X}(\text{SL}(2))) \to \mathcal{X}(\text{SL}_{2g+1}).$$

Let $e = c(1) \in \mathcal{X}(\text{SL}(2))$, where $1 \in \Pi_{\text{disc}}(\text{PGL}(2))$ is the trivial representation. Then $e_p = \text{diag}(p^{1/2}, p^{-1/2})$ and $e_\infty = (1/2, -1/2)$.

**Theorem 3.1.** (Arthur’s Endoscopic Classification [CR, Theorem 3.12],[A, Theorem 1.5.2]) Given $\pi \in \Pi_{\text{disc}}(G)$, there is $\psi(\pi) \in \Psi_{\text{glob}}(G)$ (the global Arthur parameter of $\pi$) such that

$$\text{St}(c(\pi)) = \rho_\psi(\pi) \prod_{i=1}^{k} c(\pi_i) \times e).$$

As an example, if $\pi_F$ is the cuspidal automorphic representation of $\text{PGL}(2)(\mathbb{A})$ associated with a normalised, cuspidal Hecke eigenform $f = \sum_{n=1}^{\infty} a_n q^n$ of weight $2k-2$ for $\text{SL}(2, \mathbb{Z})$, with $k$ even, if $F$, a cuspidal form of weight $k$ for $\text{Sp}(2, \mathbb{Z})$, is the Saito-Kurokawa lift of $f$, and if $\pi_F$ is the associated cuspidal automorphic representation of $\text{Sp}(2, \mathbb{A})$, then $\psi(\pi_F) = \pi_F[2] \otimes [1]$, with $c_\infty(\pi_F) = \text{diag}(k-1, k-2, 0, 2-k, 1-k), c_p(\pi_F) = \text{diag}(\alpha_p^1, \alpha_p^{-1/2}, 1, \alpha_p^{-1/2})$ and standard $L$-function

$$L(s, F; St) = \prod_{i=1}^{n}(\det(I - St(c_F(\pi_F))p^{-s}))^{-1} = \zeta(s)L(f, s+(k-1))L(f, s+(k-2)).$$

At this point we should say a little more about the relation between Siegel modular forms and automorphic representations. Asgari and Schmidt [AS] describe how to get a cuspidal automorphic representation $\pi'_F$ of $\text{PGSp}_g(\mathbb{A})$, holomorphic discrete series at $\infty$, from a Hecke eigenform $F$ in $S_k(\text{Sp}_g(\mathbb{Z}))$, with $k \geq g+1$, and something similar works for vector-valued forms [T, §5.2]. From this $\pi'_F$ one can get a cuspidal automorphic representation $\pi_F$ of $\text{Sp}_g(\mathbb{A})$, whose Satake parameters are obtained from those of $\pi'_F$ by applying the 2-to-1 covering map from $\text{Spin}(g+1, g)$ to $\text{SO}(g+1, g)$. Conversely, given $\pi \in \Pi_{\text{disc}}(\text{Sp}_g)$ with $c_\infty(\pi) = \text{diag}(k-1, \ldots, k-g, 0, g-k, \ldots, 1-k)$ and $\pi_\infty$ holomorphic discrete series, it comes from some $\pi' \in \Pi_{\text{disc}}(\text{PGSp}_g(\mathbb{A}))$ (by [CR, Proposition 4.7]), which is actually in $\Pi_{\text{cusp}}(\text{PGSp}_g(\mathbb{A}))$ (by [T, Remark 5.2.3]). This is then of the form $\pi'_F$ for some Hecke eigenform (for the $T(p)$ as well as the $T_i(p^2)$) $F \in S_k(\text{Sp}_g(\mathbb{Z}))$, as explained in [T, §5.2].

### 4. Arthur’s multiplicity formula

Closely related to $\rho_\psi$ above is

$$r_\psi : \prod_{i=1}^{k} (\hat{G}^{n_i} \times \text{SL}(2)) \to \hat{G} = \text{SO}(g+1, g).$$

Then $\text{St} \circ r_\psi$ is a direct sum $\bigoplus_{i=1}^{k} V_i$, where $V_i$ is an irreducible $n_i$-dimensional representation of $\hat{G}^{n_i} \times \text{SL}(2)$. Following [CR, §3.20], let $C_\psi$ be the centraliser of $\text{im}(r_\psi)$ in $\hat{G}$. This is an elementary abelian 2-group generated by $Z(\hat{G})$ and
elements $s_i$ for those $i$ such that $n_i$ is even, where $\text{St}(s_i)$ acts as $-1$ on $V_i$, and as $+1$ on $V_j$ for all $j \neq i$.

Arthur [A] defined a character $\epsilon_\psi : C_\psi \rightarrow \{\pm 1\}$, where $\epsilon_\psi$ is trivial on $Z(\hat{G})$ and

$$\epsilon_\psi(s_i) = \prod_{j \neq i} \epsilon(\pi_i \times \pi_j)^{\min(d_i, d_j)},$$

$\epsilon(\pi_i \times \pi_j) = \pm 1$ being the global epsilon factor appearing in the functional equation of $L(s, \pi_i \times \pi_j)$, which in our case, where $\pi_i \times \pi_j$ will be unramified at all finite primes, is just the local factor $\epsilon_\infty(\pi_i \times \pi_j)$.

Given $\pi \in \Pi(G)$ such that $L(\pi) = \psi \in \Psi_{\text{alg}}$ (a certain subset of $\Psi_{\text{glob}}(G)$, see [CR, Definition 3.15]), we can ask whether $\pi$ actually occurs in $\Pi_{\text{disc}}(G)$. Arthur’s multiplicity conjecture answers this question. The answer depends on comparing $\epsilon_\psi$ with another character which depends on how all the $\pi_p$ and $\pi_\infty$ sit in their $L$-packets. Since all the $\pi_p$ are unramified, their $L$-packets are trivial, i.e. they are uniquely determined up to isomorphism by their $c_p(\pi)$. Therefore we only need consider $\pi_\infty$, which we want to be the holomorphic discrete series representation within its $L$-packet. There is an associated Shelstad parameter $\chi_{\text{hol}} : C_{\psi_\infty} \rightarrow \mathbb{C}^\times$, where $C_{\psi_\infty}$ is a certain group which can be viewed as a 2-torsion subgroup of $\hat{\mathbb{C}}$.

5. Application to various lifts

All the propositions in this section are conditional upon Arthur’s multiplicity conjecture.

5.1. Ikeda lifts. For $k, g \geq 2$ even, and $f \in S_{2k-g}(\text{SL}(2, \mathbb{Z}))$ a Hecke eigenform, let $\pi_f$ be the associated cuspidal, automorphic representation of $\text{PGL}(2)(\mathbb{A})$, and consider $\psi_f(\gamma) \oplus [1] \in \Psi_{\text{alg}}$.

**Proposition 5.1.** There exists $\pi \in \Pi_{\text{disc}}(\text{Sp}_g)$ such that $\psi(\pi) = \psi_f(\gamma) \oplus [1]$.

**Proof.** Since $n_1 = 2g$ is even, but $n_2 = 1$ is odd, $C_{\psi}$ is generated by $Z(\hat{G})$ and $s_1 =: s_f$. We have $\epsilon_\psi(s_f) = \epsilon_\infty(\pi_f \times 1)^2 = \epsilon_\infty(\pi_f)$. Note that $\epsilon_\infty(\pi_f) = \text{diag}(\frac{2k-g-1}{2}, \frac{1-g-2k}{2})$. The associated motive (twisted to have weight 0) would have Hodge type $\{(p, q), (q, p)\}$, with $p = \frac{1-g-2k}{2}$ and $q = \frac{2k-g-1}{2}$. Putting this in the formula $T^g \cdot \pi_f$ in the table in [De, §5.3], we recover the well-known $\epsilon_\infty(\pi_f) = \epsilon^{2k-g} = (-1)^{k-1} = (-1)^{g/2}$. Of course we would have to make a half-integral twist to really have a motive, with integral Hodge weights, but since we are only interested in the difference $q - p$, we can ignore this.

On the other hand $\chi_{\text{hol}} = \hat{\epsilon}_1 + \ldots + \hat{\epsilon}_{g-1}$ (odd subscripts), which has $\frac{g}{2}$ terms, and $s_f = \text{diag}(-1, \ldots, -1, 1, -1, \ldots, -1)$, so $\chi_{\text{hol}}(s_f) = (-1)^{g/2}$. Since this is the same as $\epsilon_\psi(s_f)$, $\pi$ exists. \qed
Note that \( c_\infty(\pi) = \text{diag}(k - 1, k - 2, \ldots, k - g, 0, g - k, \ldots, 2 - k, 1 - k) \) matches \( c_\infty(\pi_F) \), where \( \pi_F \) is the automorphic representation of \( \text{Sp}_2(\mathbb{A}) \) associated with a cuspidal Hecke eigenform \( F \in S_k(\text{Sp}_g(\mathbb{Z})) \), and since \( \pi_\infty \) is holomorphic discrete series, \( \pi \) is of the form \( \pi_F \). From \( \psi(\pi_F) \) we can read off the standard \( L \)-function \( L(s, F, \text{St}) = \zeta(s) \prod_{i=1}^{\dim F} L(f, s + (k - i)) \), and we recognise \( F \) as the Ikeda lift of \( f \) \([\text{Ik}1]\).

5.2. Standard lifts. Let \( k, g, f \) be as in the previous section, and let \( F \) be a cuspidal Hecke eigenform for \( \text{Sp}_2(\mathbb{Z}) \), of weight \( \det^c \otimes \text{Sym}^j(\mathbb{C}^2) \), with \( (\kappa, j) = (k - g + 2, g - 2) \) (so we must impose \( k > g - 2 \)). To \( F \) we associate an automorphic representation \( \pi_F^a \) of \( \text{Sp}_2(\mathbb{A}) \), with \( \pi_\infty(\pi_F) = \text{diag}(j + \kappa - 1, \kappa - 2, 0, 2 - \kappa, 1 - j - \kappa) = \text{diag}(k - 1, k - g - 0, g - k, 1 - k) \). To get \( \text{diag}(k - 1, k - 2, \ldots, k - g, \ldots, 2 - k, 1 - k) \) (as seen in the previous section) from \( \text{diag}(k - 1, k - g, 0, g - k, 1 - k) \), we need to fill in the gaps using \( (g - 2) \) copies of \( \pi_\infty(\pi_F) = \text{diag}(\frac{2k-g-1}{2}, \frac{1-g-2k}{2}) \), shifted to left and right. So we consider \( \psi = \pi_F^a \oplus \pi_f[g - 2] \in \Psi_{\text{disc}} \). Note that we have abused notation somewhat; \( \pi_F^a \) is a representation of \( \text{Sp}_2(\mathbb{A}) \), but we are using the same notation for its lift to \( \text{PGL}(5)(\mathbb{A}) \), via \( \text{St} : \text{SO}(3, 2) \rightarrow \text{SL}(5) \). We must insist that we are in a situation where this lift is cuspidal, so we must exclude the case where \( g = 2 \) and \( F \) is a Saito-Kurokawa lift. (Similar remarks apply in subsequent sections.) In fact, we may as well exclude the case \( g = 2 \), in which \( F \) is already scalar-valued, and \( \pi \) below would be just the same as \( \pi_F^a \).

**Proposition 5.2.** There exists \( \pi \in \Pi_{\text{disc}}(\text{Sp}_2(\mathbb{A})) \) such that \( \psi(\pi) = \pi_F^a \oplus \pi_f[g - 2] \).

**Proof.** Since \( n_1 = 5 \) is odd, but \( n_2 = 2(g - 2) \) is even, \( C_\psi \) is generated by \( Z(\mathbb{G}) \) and \( s_2 = s_f \). We have \( \epsilon_\psi(s_f) = \epsilon(\pi_f \times \pi_F^a)^{1} = \epsilon(\pi_f \times \pi_F^a) \). Since \( \epsilon_\infty(\pi_f) = \text{diag}(\frac{2k-g-1}{2}, \frac{1-g-2k}{2}) \) and \( \epsilon_\infty(\pi_F) = \text{diag}(k - 1, k - g, 0, g - k, 1 - k) \), the associated motive (twisted to have weight \( 0 \)) would have Hodge type \( a \) union of \( 1 \) of the \( \pi_f \) scalar-valued, and \( \epsilon_\infty(\pi_F) \). We have \( \epsilon_f(\pi_f) = \epsilon(\pi_f \times \pi_F^a) \). Putting this in the formula \( i_{\nu - n + 1} = i_{2\nu + 1} \), we find that

\[
\epsilon_\infty(\pi_f \times \pi_F^a) = i^{4k-g-2+4k-3g+2k-g+g} = i^{g+2} = (-1)^{(g/2)+1}.
\]

On the other hand \( s_f = \text{diag}(1, -1, \ldots, -1, 1, 1, 1 - 1, \ldots, -1, 1) \). In the left half, \( \frac{g}{2} - 1 \) of the \( -1 \)s are in odd position, so \( \chi_{\text{hol}}(s_f) = (-1)^{(g/2)+1} \). Since this is the same as \( \epsilon_\psi(s_f) \), \( \pi \) exists.

As already noted, \( c_\infty(\pi) = \text{diag}(k - 1, k - 2, \ldots, k - g, 0, g - k, \ldots, 2 - k, 1 - k) \), as in the previous section \( \pi = \pi_G \) for some cuspidal Hecke eigenform \( G \in S_k(\text{Sp}_g(\mathbb{Z})) \). This time \( L(s, G, \text{St}) = L(s, F, \text{St}) \prod_{i=1}^{\dim F} L(f, s + (k - g + i)) \). The existence of such a \( G \) is precisely \([\text{Ib}, \text{Conjecture 3.2}]\).

5.3. Spinor lifts. Now \( k, g \geq 2 \) even, \( f \in S_{2k-g}(\text{SL}(2, \mathbb{Z})) \), and \( F \) is a cuspidal Hecke eigenform for \( \text{Sp}_2(\mathbb{Z}) \), of weight \( \det^c \otimes \text{Sym}^j(\mathbb{C}^2) \), with \( (\kappa, j) = (r + 1, 2k - g - 1 - r) \) (so we impose \( k > \frac{g}{2} + r + 1 \)), for some fixed odd \( r \) with \( \frac{g}{2} + 1 \leq r < g \). To \( F \) we associate an automorphic representation \( \pi_F^{\text{spin}} \) of \( \text{PGSp}_2(\mathbb{A}) \cong \text{SO}(3, 2)(\mathbb{A}) \), with

\[
c_\infty(\pi_F^{\text{spin}}) = \text{diag} \left( \frac{j + 2\kappa - 3}{2}, \frac{j + 1}{2}, -\frac{j + 1}{2}, -\frac{j + 2\kappa - 3}{2} \right).
\]
Then

\[ c_\infty(\pi_F^{\text{spin}}[g + 1 - r]) \]

= \text{diag}(k-1,\ldots,k+r-g-1,k-r,\ldots,k-g,0,g-k,\ldots,2,k-1) ,

where the dots denote the spinor. In the special case \( \pi_f[2r - g - 2] \), then to put 0 in the middle we use [1]. Thus

\[ c_\infty(\pi_F^{\text{spin}}[g + 1 - r] \oplus \pi_f[2r - g - 2] \oplus [1]) \]

= \text{diag}(k-1,k-2,\ldots,k-g,0,g-k,\ldots,2,k-1) .

Note that since \( r > 2 \) and \( j > 0 \), there are no entries in \( c_\infty(\pi_F^{\text{spin}}) \) differing by 1, so in the Arthur parameter of \( \pi_F^{\text{spin}} \), all \( d_i = 1 \). The possibility that \( \pi_F^{\text{spin}} \) is endoscopic is ruled out, since there are no holomorphic Yoshida lifts at level 1. Hence the lift of \( \pi_F^{\text{spin}} \) to \( \text{PGL}(4) \), which is what is really meant above by \( \pi_F^{\text{spin}} \), must be cuspidal, as desired.

**Proposition 5.3.** If \( 4 \mid g \), there exists \( \pi \in \Pi_{\text{disc}}(\text{Sp}_g) \) such that \( \psi(\pi) = \pi_F^{\text{spin}}[g + 1 - r] \oplus \pi_f[2r - g - 2] \oplus [1] \).

**Proof.** This time \( n_1 = 4(g + 1 - r) \) and \( n_2 = 2(2r - g - 2) \) are even, while \( n_3 = 1 \) is odd, so we must consider \( s_1 =: s_F \) and \( s_2 =: s_f \). Since \( \text{G}_f \) and \( \text{G}_f^{\text{spin}} \) are both symplectic, it follows from a theorem of Arthur (see [CR, §3.20]) that

\[ \epsilon(\pi_f \times \pi_F^{\text{spin}}) = 1. \]

Hence \( \epsilon(\pi_f) = \epsilon_\infty(\pi_f \times 1)^1 = \epsilon_\infty(\pi_f) = (-1)^{g/2} \) as before, and likewise \( \epsilon(\pi_F^{\text{spin}}) = \epsilon_\infty(\pi_F^{\text{spin}}) = i(2k-g-r+1)(2k-g+r-1) = (-1)^{g/2} \).

\[ s_f = \text{diag}(1,\ldots,1,1,\ldots,1,1,\ldots,1,1,\ldots,1), \]

and on the left side the number of -1s in odd position is \( r - \frac{g}{2} - 1 \), so \( \chi_{\text{hol}}(s_f) = (-1)^{-(g/2)-1} = (-1)^{g/2} \), since \( r \) is odd.

\[ s_F = \text{diag}(-1,\ldots,-1,1,\ldots,1,1,\ldots,1,1,\ldots,1), \]

and on the left side the number of -1s in odd position is \( g + 1 - r \), which is even, so \( \chi_{\text{hol}}(s_F) = 1 \). Thus, though \( \chi_{\text{hol}}(s_f) = \epsilon(\psi(s_f)) \), for \( \chi_{\text{hol}}(s_F) = \epsilon(\psi(s_F)) \) we need the condition \( 4 \mid g \). \( \square \)

As already noted, \( c_\infty(\pi) = \text{diag}(k-1,k-2,\ldots,k-g,0,g-k,\ldots,2,k-1) \), so as before, \( \pi = \pi_G \) for some cuspidal Hecke eigenform \( G \in \mathcal{S}_l(\text{Sp}_g(\mathbb{Z})) \). This time

\[ L(s, G, St) = \zeta(s) \prod_{i=1}^{r-1} L(s-i+(g-r+2)/2,F,\text{spin}) \prod_{i=1}^{r} L(f,s+(k-r+i)) \]

where the spinor \( L \)-function is in its automorphic normalisation, centred at \( s = 1/2 \). In the special case \( r = \frac{g}{2} + 1 \) (in which case \( f \) does not actually appear), the existence of such a \( G \) is precisely [Ib, Conjecture 3.1].
5.4. Ikeda-Miyawaki lifts. Consider Hecke eigenforms \( f \in S_{2k}(\text{SL}(2, \mathbb{Z})) \), \( h \in S_{k+n+1}(\text{SL}(2, \mathbb{Z})) \), where \( k+n+1 \) is even. Let \( \pi_f \) be the associated cuspidal, automorphic representation of \( \text{PGL}(2)(\mathbb{A}) \), and \( \pi_h^{st} \) the cuspidal automorphic representation of \( \text{Sp}_4(\mathbb{A}) \) associated with \( h \). Recall that \( c_p(\pi_h^{st}) = \text{diag}(\alpha^2_p, 1, \alpha_p^{-2}) \in \text{SO}(2, 1)(\mathbb{C}) \) (where \( a_p(h) = \frac{p^{(k+n)/2}(\alpha_p + \alpha_p^{-1})}{p} \)), and \( c_\infty(\pi_h^{st}) = \text{diag}(k+n, 0, -k-n) \). Since \( c_\infty(\pi_f) = \text{diag}(\frac{k+n-1}{2}, \frac{-k-1}{2}) \), we see that \( c_\infty(\pi_h^{st} \oplus \pi_f[2n]) = \text{diag}(k+n, n, \ldots, k-n, 0, \ldots, -n-k) \), where the dots denote unbroken sequences of consecutive integers. This is of the form \( \text{diag}(\kappa-1, \kappa-2, \ldots, \kappa-g, 0, g-\kappa, \ldots, 2-\kappa, 1-\kappa) \), where \( \kappa = k+n+1 \) and \( g = 2n+1 \).

**Proposition 5.4.** There exists \( \pi \in \Pi_{\text{disc}}(\text{Sp}_{2n+1}) \) such that \( \psi(\pi) = \pi_h^{st} \oplus \pi_f[2n] \).

**Proof.** Since \( n_1 = 3 \) is odd, while \( n_2 = 4n \) is even, we consider \( s_2 =: s_f \). First, \( \epsilon_\psi(s_f) = c_\infty(\pi_h^{st} \times \pi_f) \). The associated motive (twisted to have weight 0) would have Hodge type \( \text{a union of} \left\{ (-q, q), (q, -q) \right\} \), where \( 2q \) runs through \( \{2k-1+2(2k+n) = 4k+2n-1, 2k-1, 2n+1 \} \). Putting this in the formula \( i^q = i^{2q+1} \), we find that \( c_\infty(\pi_f) = i^{2k+2} = (-1)^{k+1} \).

Now \( s_f = \text{diag}(1, -1, \ldots, -1, 1, 1) \), and in the left half, \( n \) of the \(-1\)s are in odd position, so \( \chi_{\text{hol}}(s_f) = (-1)^n \), which is the same as \( (-1)^{k+1} \), since \( n+k+1 \) is even.

As already noted, \( c_\infty(\pi) = \text{diag}(k+n, \ldots, k-n, 0, n-k, \ldots, -n-k) \), so \( \pi = \pi_G \) for some cuspidal Hecke eigenform \( G \in S_{k+n+1}(\text{Sp}_{2n+1}(\mathbb{Z})) \). Also \( L(s, G, St) = L(s, h, St) \prod_{n=1}^{2n} L(f, s+(k-n-1+i)) \), and we recognise \( G \) as a lift whose existence was conjectured by Miyawaki and proved by Ikeda [Miy, Ik2].

5.5. Lifts from genus 3 and 1. Let \( f \) be as in the previous section, with \( k+n+1 \) still even. Let \( F \) be a vector-valued cuspidal Hecke eigenform of genus 3 such that if \( \pi_F^{st} \) is the associated automorphic representation of \( \text{Sp}_4(\mathbb{A}) \), then \( c_\infty(\pi_F^{st}) = \text{diag}(k+n, k+n-1, -k-n, 0, n-k, -n+k+1, -n-k) \). In the language of [BFvdG, §4.1.7], \( (a, b, c) = (k+n-3, k+n+3, k-n-1) \). To fill in the gaps of length \( 2n-2 \), we consider \( \psi = \pi_F^{st} \oplus \pi_f[2n-2] \). We may as well exclude the case \( n = 1 \), in which \( F \) is already scalar-valued and \( \pi \) below would be just the same as \( \pi_F^{st} \).

**Proposition 5.5.** There exists \( \pi \in \Pi_{\text{disc}}(\text{Sp}_{2n+1}) \) such that \( \psi(\pi) = \pi_F^{st} \oplus \pi_f[2n-2] \).

**Proof.** Since \( n_1 = 7 \) is odd, while \( n_2 = 4n-4 \) is even, we consider \( s_2 =: s_f \).

\[
\epsilon_\psi(s_f) = c_\infty(\pi_F^{st} \times \pi_f) = i^{(4k+2n)+(4k+2n-2)+(4k-2n)+2k+2n+2(2n+2)+2n} = i^{2k} = (-1)^k.
\]

\[
s_f = \text{diag}(1, -1, \ldots, -1, 1, 0, 1, -1, \ldots, -1, 1, 1),
\]

with \( n-1 \) of \(-1\)s in the left half in odd position, so \( \chi_{\text{hol}}(s_f) = (-1)^{n-1} \), which is the same as \( (-1)^k \), since \( k+n+1 \) is even.

As before, \( \pi = \pi_G \) for some cuspidal Hecke eigenform \( G \in S_{k+n+1}(\text{Sp}_{2n+1}(\mathbb{Z})) \). We read off \( L(s, G, St) = L(s, F, St) \prod_{n=1}^{2n-2} L(f, s+k-n+i) \).
5.6. Lifts from genus 1, 2 and 1. As in §5.4, consider Hecke eigenforms \( f \in S_{2n}(\text{SL}(2, \mathbb{Z})) \), \( h \in S_{k+n+1}(\text{SL}(2, \mathbb{Z})) \), where \( k + n + 1 \) is even. Let \( \pi_f \) be the associated cuspidal, automorphic representation of \( \text{PGL}(2)(\mathbb{A}) \), and \( \pi^\text{st}_h \) the cuspidal automorphic representation of \( \text{Sp}_1(\mathbb{A}) = \text{SL}_2(\mathbb{A}) \) associated with \( h \). Let \( F \) be a cuspidal Hecke eigenform for \( \text{Sp}_1(\mathbb{A}) \), of weight \( \text{det}^r \otimes \text{Sym}^2(\mathbb{C}^2) \), with \((\kappa, j) = (r + 1, 2k - 1 - r)\), for some fixed odd \( r \) with \( n + 1 \leq r \leq 2n - 1 \). To \( F \) we associate an automorphic representation \( \pi^\text{spin}_F \) of \( \text{PGSp}_2(\mathbb{A}) \simeq \text{SO}(3, 2)(\mathbb{A}) \), with

\[
c_{\infty}(\pi^\text{spin}_F) = \text{diag} \left( \frac{j + 2k - 3}{2}, \frac{j + 1}{2}, \frac{j + 1}{2}, -\frac{j + 2k - 3}{2} \right),
\]

\[
= \text{diag} \left( \frac{2k + r - 2}{2}, \frac{2k - r}{2}, \frac{2k - r}{2}, -\frac{2k + r - 2}{2} \right).
\]

Then

\[
c_{\infty}(\pi^\text{spin}_h[2n + 1 - r]) = \text{diag}(k + n - 1, \ldots, k + r - n - 1, k + n - r, \ldots, k - n, k, \ldots, r - n - k, 1 + n - r - k, \ldots, 1 - k - n),
\]

where the dots denote unbroken sequences of consecutive integers. To fill in the gaps we use \( \pi_f[2r - 2n - 2] \), and we also add \( c_{\infty}(\pi^\text{st}_h) = \text{diag}(k + n, 0, -n - k) \). Thus

\[
c_{\infty}(\pi^\text{st}_h \oplus \pi^\text{spin}_F[2n + 1 - r] \oplus \pi_f[2r - 2n - 2]) = \text{diag}(k + n, k + n - 1, \ldots, k - n, 0, n - k, \ldots, 1 - n - k, -n - k).
\]

**Proposition 5.6.** There exists \( \pi \in \Pi_{\text{disc}}(\text{Sp}_{2n+1}) \) such that \( \psi(\pi) = \pi^\text{st}_h \oplus \pi^\text{spin}_F[2n + 1 - r] \oplus \pi_f[2r - 2n - 2] \).

**Proof.** This time \( n_2 = 4(2n + 1 - r) \) and \( n_3 = 2(2r - 2n - 2) \) are even, while \( n_1 = 3 \) is odd, so we must consider \( s_2 =: s_F \) and \( s_3 =: s_f \). Since \( G^\text{st}_F \) and \( G^\text{spin}_F \) are both symplectic, it follows from a theorem of Arthur (see [CR, §3.20]) that \( \epsilon(\pi_f \times \pi^\text{spin}_F) = 1 \). Hence

\[
\epsilon_{\psi}(s_f) = \epsilon_{\infty}(\pi_f \times \pi^\text{st}_h)^1 = \text{exp}^{-2k+2(2n+2)+(4k+4n)} = (-1)^{k+1},
\]

and likewise

\[
\epsilon_{\psi}(s_F) = \epsilon_{\infty}(\pi^\text{spin}_F \times \pi^\text{st}_h)^1 = \text{exp}^{-2k+r+2(2k-r+2)+(4k+4n)} = (-1)^{k+1},
\]

\[
s_f = \text{diag}(1, \ldots, 1, -1, \ldots, -1, 1, \ldots, 1, -1, \ldots, -1, 1, \ldots, 1),
\]

\[
s_F = \text{diag}(-1, \ldots, -1, 1, \ldots, 1, -1, \ldots, -1, 1, \ldots, 1, -1, \ldots, -1, 1, \ldots, 1),
\]

and on the left side the number of \(-1\)s in odd position is \( r - n - 1 \), so \( \chi_{\text{hol}}(s_f) = (-1)^{r-n-1} = (-1)^n \), since \( r \) is odd. This is the same as \((-1)^{k+1} \), since \( n + k + 1 \) is even.

\[
s_f = \text{diag}(-1, \ldots, -1, 1, \ldots, 1, -1, \ldots, -1, 1, \ldots, 1, -1, \ldots, -1, 1, \ldots, 1),
\]

\[
s_F = \text{diag}(-1, \ldots, -1, 1, \ldots, 1, -1, \ldots, -1, 1, \ldots, 1, -1, \ldots, -1, 1, \ldots, 1),
\]

and on the left side the number of \(-1\)s in odd position is \( 2n + 1 - r \), which is even, so \( \chi_{\text{hol}}(s_F) = 1 \). \( \square \)
We have $\pi = \pi_G$ for some cuspidal Hecke eigenform $G \in S_{k+n+1}(\text{Sp}_{2n+1}(\mathbb{Z}))$, and we get $L(s, G, St)$

$$L(s, h, St) \prod_{i=1}^{2n+r-1} L(s + \frac{2n-r}{2} + 1 - i, F, \text{spin}) \prod_{j=1}^{2r-2n-2} L(f, s + k + n - r + j).$$

Note that in the case $r = n + 1$, $f$ does not appear.

6. Congruences between lifts and “non-lifts”

6.1. Congruences between Ikeda lifts and non-Ikeda lifts. The following is Theorem 4.7 of [Ka1]. The proof makes use of the proof by Katsurada and Kawamura [KK] of a conjecture of Ikeda on the Petersson norm of his lift. The normalised $L$-values $L_{\text{alg}}(f, k)$ and $L_{\text{alg}}(2i + 1, f, St)$ are obtained from $L(f, k)$ and $L(2i + 1, f, St)$ by dividing by suitably normalised Deligne periods, as explained in [BD, §4]. For $L_{\text{alg}}(f, k)$, the Deligne period is as constructed in [Ka1, §4], using parabolic cohomology with integral coefficients. (Since $q > 2k$, we may ignore various factorials of small numbers.) For $L_{\text{alg}}(2i + 1, f, St)$ it is essentially a product $\Omega^+\Omega^-$ of normalised Deligne periods for $L(f, s)$ [Du, Lemma 5.1], but given the condition (2) below, this is as good as the $(f, f)$ used by Katsurada (see condition (3) in [Kal, Theorem 4.7]).

**Theorem 6.1.** For $k, g \geq 2$ even, and $f \in S_{2k-g}(\text{SL}(2, \mathbb{Z}))$ a Hecke eigenform, let $F \in S_k(\text{Sp}_g(\mathbb{Z}))$ be the Ikeda lift, as in §5.1 above. Suppose that $k \geq 2g + 4$ and that $q > 2k$ is a prime number such that, for some divisor $q | q$ in a sufficiently large number field,

$$\text{ord}_q(L_{\text{alg}}(f, k) \prod_{i=1}^{(g/2)-1} L_{\text{alg}}(2i + 1, f, St)) > 0.$$

Suppose further that

1. for some even integer $t$ with $k + 2 \leq t \leq 2k - 2g - 2$, and some fundamental discriminant $D$ with $(-1)^{t/2}D > 0$,

$$\text{ord}_q\left(\frac{\zeta(t + g - k)}{\zeta(t + g + k)} \prod_{i=1}^{g} L_{\text{alg}}(f, t + i - 1) L_{\text{alg}}(f, (k - 2g)/2, \chi_D)D\right) = 0,$$

where $\chi_D$ is the associated quadratic character, and the Dirichlet $L$-value is normalised as in [Ka1];

2. there is not a congruence mod $q$ of Hecke eigenvalues between $f$ and another Hecke eigenform in $S_{2k-g}(\text{SL}(2, \mathbb{Z}))$;

3. if $g > 2$, $q \nmid \prod_{p < 2k-g \text{ prime}} (1 + p + p^2 + \ldots + p^{g-1}).$

Then there exists a Hecke eigenform $G \in S_k(\text{Sp}_g(\mathbb{Z}))$, not the Ikeda lift of any Hecke eigenform $h \in S_{2k-g}(\text{SL}(2, \mathbb{Z}))$, such that for any prime $p$, corresponding Hecke eigenvalues for $F$ and $G$, for all the Hecke operators $T(p)$ and $T_i(p^2) \ (1 \leq i \leq g)$, are congruent mod $q$.

Ikeda proved only that $F$ is a Hecke eigenform for the $T_i(p^2)$ (defined in [Ka1, §2]), which generate a Hecke algebra associated with the pair $(\text{Sp}_g(\mathbb{Q}_p), \text{Sp}_g(\mathbb{Z}_p))$, but Katsurada [Ka1, Proposition 4.1] extended this to $T(p)$, which with the $T_i(p^2)$ generates a Hecke algebra associated with $(\text{GSp}_g(\mathbb{Q}_p), \text{GSp}_g(\mathbb{Z}_p))$. (See also the
final paragraph of §3 above.) If we ignore the $T(p)$ then the congruence in the theorem is equivalent to a congruence (for all $p$) of Satake parameters

$$c_p(\pi_F) \equiv c_p(\pi_G) \pmod{q},$$

(or strictly speaking $p^{k_0-g(g+1)/2}c_p(\pi_F) \equiv p^{k_0-g(g+1)/2}c_p(\pi_G) \pmod{q}$), with

$$c_p(\pi_F) = \text{diag}(\alpha_{1,F}, \ldots, \alpha_{g,F}, 1, \alpha_{g,F}^{-1}, \ldots, \alpha_{1,F}^{-1}) \in \hat{T}(\mathbb{C}),$$

and likewise for $G$. We should interpret the congruence as being between $c_p(\pi_F)$ and some element in the orbit of $c_p(\pi_G)$ under the action of a Weyl group that can permute the indices $1, \ldots, g$ and switch pairs $\alpha_{i,F}$ and $\alpha_{i,F}^{-1}$, in fact $c_p(\pi_F)$ really should be thought of as a conjugacy class in $\hat{G}(\mathbb{C})$, represented by the above element of $\hat{T}(\mathbb{C})$. To include $T(p)$ as well, we would need to consider also $\alpha_{0,F}$ with

$$\alpha_{0,F}^2 \prod_{i=1}^{n-1} \alpha_{i,F} = 1,$$

for each $p$.

6.2. Congruences between Ikeda-Miyawaki lifts and non-Ikeda-Miyawaki lifts. The congruences are inspired by a conjecture of Ikeda on the Petersson norm of the Ikeda-Miyawaki lift. The following is taken from Conjecture B and Problem B’ of [IKPY], which may be formulated, given $[H, vdG]$, in theory we take it as in [BD, §4]. Ibukiyama, Katsurada, Poor and Yuen use a practical substitute when they prove an instance of the congruence in [IKPY, §5].

**Conjecture 6.2.** For Hecke eigenforms $f \in S_{2k}(SL(2,\mathbb{Z}))$, $h \in S_{k+n+1}(SL(2,\mathbb{Z}))$, where $k+n+1$ is even, let $F \in S_{k+n+1}(Sp_{2n+1}(\mathbb{Z}))$ be the Ikeda-Miyawaki lift, as in §5.4. Suppose that $q > 2k+2n-2$ is a prime number such that, for some divisor $q \mid q$ in a sufficiently large number field,

$$\text{ord}_q(L_{\text{alg}}(f \otimes \text{Sym}^2 h, 2k+2n) \prod_{i=1}^{n-1} L_{\text{alg}}(2i+1, f, St)) > 0.$$

Then there exists a Hecke eigenform $G \in S_{k+n+1}(Sp_{2n+1}(\mathbb{Z}))$, not the Ikeda-Miyawaki lift of any Hecke eigenforms $f' \in S_{2k}(SL(2,\mathbb{Z}))$, $h' \in S_{k+n+1}(SL(2,\mathbb{Z}))$, such that for any prime $p$, corresponding Hecke eigenvalues for $F$ and $G$, for all the Hecke operators $T(p)$ and $T_i(p^2)$ ($1 \leq i \leq g$), are congruent mod $q$.

Remarks about congruences of Satake parameters, similar to the previous subsection, apply.

7. Accounting for some of the congruences

7.1. Ikeda lifts and standard lifts: $L_{\text{alg}}(f, k)$. We have $2k-g = j + 2k-2, k = j + k$, if $(\kappa, j) = (k+2-g, g-2)$, in agreement with §5.2 above. Harder’s conjecture [H, vdG] may be formulated, given $q \mid q$ with $q > 2k-g$ and $\text{ord}_q(L_{\text{alg}}(f, k)) > 0$, as the existence of a Hecke eigenform $F$ for $Sp_2(\mathbb{Z})$, of weight $\det^k \otimes \text{Sym}^2(\mathbb{C}^2)$, such that if $\pi_F^p$ is the associated automorphic representation of $Sp_2(\mathbb{A})$ then for all primes $p$,

$$c_p(\pi_F^p) \equiv \text{diag}(\alpha_p^{p(g-1)/2}, \alpha_p^{-1}p^{(g-1)/2}, 1, \alpha_p^{-1}p^{-(g-1)/2}, \alpha_p^{-1}p^{(1-g)/2}) \pmod{q},$$

where $c_p(\pi_F) = \text{diag}(\alpha_p, \alpha_p^{-1})$. The $\frac{2k+2n-2}{2} = \frac{2k+2n-2}{2}$ is what we called $s$ in [BD]. Note that if we let $\alpha_{1,F} = \alpha_p^{p^n}$, $\alpha_{2,F} = \alpha_p^{p^{-s}}$ and $\alpha_{0,F} = \alpha_p^{-1}$ (so $\alpha_{0,F}^{s} = 1$) then

$$\alpha_{0,F} + \alpha_{0,F}^{-1} + \alpha_{0,F} \alpha_{2,F} + \alpha_{0,F}\alpha_{1,F} \alpha_{2,F} = \alpha_p + \alpha_p^{-1} + p^{-s} + p^s,$$
which when scaled by $p^{(j+2k-3)/2}$ gives the familiar $a_p(f) + p^{k-2} + p^{j+k-1}$ on the right hand side of Harder’s conjecture (as a Hecke eigenvalue for $T(p)$ on an induced representation). For simplicity we actually ignore $T(p)$, and consider only the Hecke algebra generated by $T_1(p^2)$ and $T_2(p^2)$. This is because we are looking at an automorphic representation of $\text{Sp}_2(\mathbb{A})$ rather than of $\text{GSp}_2(\mathbb{A})$. In [BD, §7], we looked at Harder’s conjecture as a congruence between Hecke eigenvalues between a cuspidal automorphic representation of $\text{Sp}_2(\mathbb{A})$ and a representation induced from the Levi subgroup $(\text{GL}_1 \times \text{GL}_2)(\mathbb{A})$ of the Siegel maximal parabolic (and worked it out explicitly only for $T(p)$). Here we can either restrict to $\text{Sp}_2(\mathbb{A})$ or just consider directly $\text{Sp}_2$ with the Levi subgroup $\text{GL}_1 \times \text{SL}_2$ of its Siegel parabolic.

Now $c_p(\pi_f[g])$

\[= \text{diag}(\alpha_pp^{(g-1)/2}, \alpha_pp^{(g-3)/2}, \ldots, \alpha_pp^{(1-g)/2}, \alpha_p^{-1}p^{(g-1)/2}, \ldots, \alpha_p^{-1}p^{(1-g)/2}),\]

and

\[c_p(\pi_f[g-2]) = \text{diag}(\alpha_pp^{(g-3)/2}, \ldots, \alpha_pp^{(3-g)/2}, \alpha_p^{-1}p^{(g-3)/2}, \ldots, \alpha_p^{-1}p^{(3-g)/2}),\]

so the congruence can be read as

\[c_p(\pi_f^g \oplus \pi_f[g-2]) \equiv c_p(\pi_f[g] \oplus [1]) \quad \text{(mod q)}.\]

Comparing with §5.1 and §5.2, we see that in the case of $q \mid L_{\text{alg}}(f,k)$, we can explain the congruence between the Ikeda lift and a non-Ikeda lift in Theorem 6.1 as a congruence between the Ikeda lift and a “standard lift” as constructed in §5.2. So the congruence in Theorem 6.1 is derived from that in Harder’s conjecture via lifting to scalar-valued large genus forms. In the excluded case $g = 2$, Harder’s conjecture is replaced by its degeneration, a congruence between a Saito-Kurokawa lift and non-lift, which does not require further lifting.

1. **Ikeda lifts and spinor lifts:** $L_{\text{alg}}(2i+1, f, St)$. If $r = 2i + 1$ then as $i$ runs from 1 to $\frac{q}{2} - 1$, $r$ runs through odd numbers from 3 to $g - 1$. We shall only be able to account for the congruence in Conjecture 6.1 if $4 \mid g$ and $\frac{q}{2} + 1 \leq r \leq g - 1$. We also require $q > 4k - 2g$. Let $(\kappa, j) = (r + 1, 2k - g - 1 - r)$, so $\kappa + j = 2k - g$ and $r = s + 1$, where $s = \kappa - 2$ as in [BD, §6]. Then a conjectural congruence of Kurokawa-Mizumoto type (instances of which were proved in [Ku, Miz, Sa, Du]) may be formulated, given $\text{ord}_q(L_{\text{alg}}(r, f, St)) > 0$, as the existence of a Hecke eigenform $F$ for $\text{Sp}_2(\mathbb{Z})$, of weight $\text{det}^\kappa \otimes \text{Sym}^j(\mathbb{C}^2)$, such that $\pi_f^{\text{spin}}$ is the associated automorphic representation of $\text{SO}(3, 2)(\mathbb{A})$ then for all primes $p$,

\[c_p(\pi_f^{\text{spin}}) \equiv \text{diag}(\alpha_pp^{s/2}, \alpha_pp^{-s/2}, \alpha_p^{-1}p^{s/2}, \alpha_p^{-1}p^{-s/2}) \quad \text{(mod q)},\]

where $c_p(\pi_f) = \text{diag}(\alpha_p, \alpha_p^{-1})$. Note that the trace of the right hand side, when scaled by $p^{(j+2k-3)/2}$, becomes the familiar $a_p(f)(1+p^{k-2})$. Recalling that $s = r-1$, this would imply that $c_p(\pi_f^{\text{spin}}[g+1 \text{ } - \text{ } r])$

\[\equiv \text{diag}(\alpha_pp^{(g-1)/2}, \ldots, \alpha_pp^{(2r-g-1)/2}, \alpha_pp^{(1-g-2r)/2}, \ldots, \alpha_p^{-1}p^{(1-g)/2}),\]

\[\text{and} \quad \alpha_p^{-1}p^{(g-1)/2}, \ldots, \alpha_p^{-1}p^{(2r-g-1)/2}, \alpha_p^{-1}p^{(1-g-2r)/2}, \ldots, \alpha_p^{-1}p^{(1-g)/2}).\]

The right hand side is the “difference” between $c_p(\pi_f[g])$ and $c_p(\pi_f[2r - g - 2])$. Thus we can read the congruence as

\[c_p(\pi_f^{\text{spin}}[g+1 \text{ } - \text{ } r] \oplus \pi_f[2r - g - 2] \oplus [1]) \equiv c_p(\pi_f[g] \oplus [1]),\]
i.e. as a congruence between the Ikeda lift and one of the “spinor lifts” constructed in §5.3. In the case of \( q \mid L_{\text{alg}}(2i + 1, f, St) \), with \( 4 \mid g, \frac{q}{2} \leq i \leq \frac{q}{2} - 1 \) and \( q > 4k - 2g \), we can explain the congruence between the Ikeda lift and a non-Ikeda lift in Theorem 6.1 (at least if we ignore \( T(p) \)) as a congruence between the Ikeda lift and a spinor lift. Thus the congruence in Theorem 6.1 is derived from that of Kurokawa-Mizumoto type via lifting to scalar-valued, large genus forms. Note that we have had to impose a stronger lower bound for \( q \).

### 7.3. Ikeda-Miyawaki lifts: \( L_{\text{alg}}(f \otimes \text{Sym}^2 h, 2k + 2n) \)

Recall that we consider Hecke eigenforms \( f \in S_{2k}(\text{SL}(2, \mathbb{Z})), h \in S_{k+n+1}(\text{SL}(2, \mathbb{Z})) \), where \( k + n + 1 \) is even. Let \( a_p(f) = p^{(2k-1)/2}(\alpha_p + \alpha_p^{-1}) \) and \( b_p(h) = p^{k+n}/2(\beta_p + \beta_p^{-1}) \). Let \( (a, b, c) = (k + n - 3, k + n - 3, k - n - 1) \), as in §5.5 above. Then \( b + c + 4 = 2k \), \( a + 4 = k + n + 1 \) (the weights of \( f \) and \( h \)), \( a + b + 6 = 2k + 2n \) and \( s := \frac{b+c+4}{2} = \frac{2n+1}{2} \). Comparing with [BD, §8, Case 2], the conjecture there (see also [BFvdG, Conjecture 10.8]), given \( \text{ord}_q(L_{\text{alg}}(f \otimes \text{Sym}^2 h, 2k + 2n)) > 0 \) with \( q > a + b + 2c + 8 = 4k \), can be formulated (ignoring \( T(p) \)) as the existence of a cuspidal Hecke eigenform \( F \) for \( \text{Sp}_q(\mathbb{Z}) \), vector-valued of type \( (a, b, c) \), such that

\[
c_p(\pi^F_0) \equiv \text{diag}(\alpha_p^s, \alpha_p^{-1} p^s, \beta_p^2, 1, \beta_p^{-2}, \alpha_p p^{-s}, \alpha_p^{-1} p^{-s}) \pmod q.
\]

To get the diagonal entries, apply the cocharacters \( f_1, f_2, f_3, 0, -f_3, -f_2, -f_1 \) to \( \gamma = -\log_p(\alpha_p) (e_1 - e_2) - \log_p(\beta_p) + s(e_1 + e_2) \) in [BD, §8], omitting \( e_0 \) since we are really dealing with \( G = \text{Sp}_3 \), \( M \simeq \text{GL}_2 \times \text{SL}_2 \).

Since \( c_p(\pi^F_0) = \text{diag}(\beta_p^2, 1, \beta_p^{-2}) \), and since \( s = \frac{2n+1}{2} \), we can read this as

\[
c_p(\pi^F_0 \oplus \pi_f[2n-2]) \equiv c_p(\pi^F_0 \oplus \pi_f[2n]) \quad (\text{mod } q),
\]

i.e. as a congruence between the Ikeda-Miyawaki lift and one of the lifts constructed in §5.5. Thus the congruence in Conjecture 6.2, between the Ikeda-Miyawaki lift and a non-Ikeda-Miyawaki lift, can be derived from the conjectured genus 3 Eisenstein congruence, via lifting to scalar-valued, large genus forms. Again, we have had to impose a stronger lower bound for \( q \). In the excluded case \( n = 1 \), the Eisenstein congruence degenerates to a congruence between an Ikeda-Miyawaki lift and a non-Ikeda-Miyawaki lift, without any further lifting.

### 7.4. Ikeda-Miyawaki lifts: \( L_{\text{alg}}(2i + 1, f, St) \)

If \( r = 2i + 1 \) then as \( i \) runs from 1 to \( n - 1 \), \( r \) runs through odd numbers from 3 to \( 2n - 1 \). We shall only be able to account for the congruence in Theorem 6.2 if \( n + 1 \leq r \leq 2n - 1 \). We also require \( q > 4k \). Let \( (\kappa, j) = (r + 1, 2k - 1 - r) \), so \( \kappa + j = 2k \) and \( r = s + 1 \), where \( s = \kappa - 2 \) as in [BD, §6]. Then a conjecture of Kurokawa-Mizumoto type, given \( \text{ord}_q(L_{\text{alg}}(r, f, St)) > 0 \), predicts the existence of a Hecke eigenform \( F \) for \( \text{Sp}_2(\mathbb{Z}) \), of weight \( \det^{\kappa} \otimes \text{Sym}^j(C^2) \), such that if \( \pi^\text{spin}_p \) is the associated automorphic representation of \( \text{SO}(3, 2)(\mathbb{A}) \) then for all primes \( p \),

\[
c_p(\pi^\text{spin}_p) \equiv \text{diag}(\alpha_p p^{s/2}, \alpha_p p^{-s/2}, \alpha_p^{-1} p^{s/2}, \alpha_p^{-1} p^{-s/2}) \pmod q,
\]

where \( c_p(\pi_f) = \text{diag}(\alpha_p, \alpha_p^{-1}) \). Recalling that \( s = r - 1 \), this would imply that

\[
c_p(\pi^\text{spin}_p[2n + 1 - 1]) \equiv \text{diag}(\alpha_p p^{(2n-1)/2}, \ldots, \alpha_p p^{(2r-2n-1)/2}, \alpha_p p^{(1+2n-2r)/2}, \ldots, \alpha_p p^{(1-2n)/2}, \alpha_p^{-1} p^{(2n-1)/2}, \ldots, \alpha_p^{-1} p^{(2r-2n-1)/2}, \alpha_p^{-1} p^{(1+2n-2r)/2}, \ldots, \alpha_p^{-1} p^{(1-2n)/2}).
\]
The right hand side is the “difference” between $c_p(\pi_f[2n])$ and $c_p(\pi_f[2r-2n-2])$. Thus we can read the congruence as

$$c_p(\pi_n^{st} + \pi_n^{pin}[2n + 1 - r] + \pi_f[2r-2n-2]) \equiv c_p(\pi_n^{st} + \pi_f[2n]),$$

i.e. as a congruence between the Ikeda-Miyawaki lift and one of the lifts constructed in §5.6. In the case of $q | L_{alg}(2i+1,f,St)$, with $\left\lfloor \frac{n}{2} \right\rfloor \leq i \leq n-1$ and $q > 4k$, we can explain the congruence between the Ikeda-Miyawaki lift and a non-Ikeda-Miyawaki lift in Conjecture 6.2 (at least if we ignore $T(p)$) as a congruence between the Ikeda-Miyawaki lift and a lift from §5.6. Thus the congruence in Conjecture 6.2 is derived from that of Kurokawa-Mizumoto type via lifting to scalar-valued, large genus forms. Again, we have had to impose a stronger lower bound for $q$.

References


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