SYMMETRIC SQUARE $L$-FUNCTIONS AND SHAFAREVICH-TATE GROUPS, II

NEIL DUMMIGAN

Abstract. We re-examine some critical values of symmetric square $L$-functions for cusp forms of level one. We construct some more of the elements of large prime order in Shafarevich-Tate groups, demanded by the Bloch-Kato conjecture. For this we use the Galois interpretation of Kurokawa-style congruences between vector-valued Siegel modular forms of genus two (cusp forms and Klingen Eisenstein series), making further use of a construction due to Urban. We must assume that certain 4-dimensional Galois representations are symplectic. Our calculations with Fourier expansions use the Eholzer-Ibukiyama generalisation of the Rankin-Cohen brackets. We also construct some elements of global torsion which should, according to the Bloch-Kato conjecture, contribute a factor to the denominator of the rightmost critical value of the standard $L$-function of the Siegel cusp form. Then we prove, under certain conditions, that the factor does occur.

1. Introduction

Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a normalised, cuspidal Hecke eigenform of weight $k'$ for $\text{SL}(2, \mathbb{Z})$. Associated to $f$ is its $L$-function

$$L_f(s) = \sum_{n=1}^{\infty} a_n n^{-s} = \prod_p \left( 1 - \alpha_p p^{-s} + p^{k'-1-2s} \right)^{-1} = \prod_p \left( 1 - \alpha_p p^{-s} \right) \left( 1 - \alpha_p p^{-s} \right)^{-1}.$$

The series converges for $\Re(s)$ sufficiently large, but there is a holomorphic continuation to the whole of $\mathbb{C}$. Likewise there is a symmetric square $L$-function

$$D_f(s) = \prod_p \left[ (1 - \alpha_p^2 p^{-s}) (1 - \beta_p p^{-s}) (1 - \alpha_p \beta_p p^{-s}) \right]^{-1}.$$

In [Du] we looked at those weights ($k' = 12, 16, 18, 20, 22, 26$) for which the dimension of the space of cusp forms is one. Using a method of Zagier [Z], we calculated the critical values of $D_f(s)$. For example, here are the values of $D_f(r + k' - 1)/(f,f)\pi^{2r+k'-1}$ for odd $r$ with $1 \leq r \leq k' - 1$, firstly when $k' = 22$. (The other critical values are related to these ones by the functional equation $\Lambda(s) = \ldots$)

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The Bloch-Kato conjecture [BK] provides a conjectural formula for these values. Large primes in the numerators ought to be the orders of elements in generalised Shafarevich-Tate groups. In [Du] we gave conditional constructions for some of these elements, in particular for the primes set in boldface in the above table. The Eisenstein series attached to and generalised by Mizumoto [M], between (the Hecke eigenvalues of) a Klingen-Eisenstein series and cusp forms [R]. In the situation dealt with by Urban, the coefficient required elements of order \( \ell = k' - 1 \) are the moduli for congruences of a type discovered by Kurokawa [Ku] and generalised by Mizumoto [M], between (the Hecke eigenvalues of) a Klingen-Eisenstein series attached to \( f \) and a cuspidal Hecke eigenform \( F \) for the Siegel modular group of genus 2. The congruence forces a 4-dimensional (mod \( \ell \)) Galois representation attached to \( F \) (by Laumon and Weissauer [L],[W], see also [Ta]) to be reducible. Exploiting this, a construction due to Urban [U] provides the required elements of order \( \ell \) in Selmer groups attached to the pre-motivic structure for \( D_f(k' - 1) \). They come from extensions inside the (mod \( \ell \)) representation, which can be made non-trivial, using the absolute irreducibility of the \( \ell \)-adic representation. This is a higher dimensional analogue of Ribet’s construction of elements in class groups of cyclotomic fields, which uses congruences between classical Eisenstein series and cusp forms [R]. In the situation dealt with by Urban, the coefficient ring is “large”, parametrising cyclotomic twists of cusp forms in a \( p \)-adic family. We adapt his construction to the simpler case of \( \ell \)-adic coefficients.

The goal of this paper is to use Urban’s construction again, but this time to get away from the right-hand edge of the critical range by using vector-valued Siegel modular forms (of genus 2). For example, here are the values of \( D_f(r + k' - 1)/(f, f) \pi^{2r+k'-1} \) for odd \( r \) with \( 1 \leq r \leq k' - 1 \) when \( k' = 16 \).

\[
\begin{align*}
\Lambda(2k - 1 - s), \text{ where } \Lambda(s) &= \Gamma_X(s - k + 2)\Gamma_C(s)D_f(s). \\
k' &= 22 \\
r &= 1 & 2^{25}/3^9.5^4.7^3.11.13.17.19 \\
3 & 2^{28}/3^{10}.5^2.7^2.11^2.17.19^2.23 \\
5 & 2^{26}.59/3^{11}.5^6.7^3.11^2.13.17.19^2.23 \\
7 & 2^{28}.58/3^{15}.5^8.7^3.11^2.13^2.19^2.23 \\
9 & 2^{23}.55/3^{16}.5^8.7^4.11^2.13^2.17.19^2.23.29 \\
11 & 2^{28}.131.593/3^{17}.5^8.7^5.11.13^2.17.19^2.23.29.31 \\
13 & 2^{28}.2436904891/3^{20}.5^{10}.7^5.11^4.13^2.17.19^2.23.29.31 \\
15 & 3^{12}.9513941/3^{22}.5^{10}.7^6.11^4.13^2.17^2.19^2.23.29.31 \\
17 & 3^{14}.545715463/3^{21}.5^{12}.7^7.11^4.13^2.17^2.19^2.23.29.31.37 \\
19 & 3^{17}.281.286397/3^{21}.5^{10}.7^7.11^4.13^3.17^3.19^2.23.29.31.37 \\
21 & 3^{17}.61.103/3^{21}.5^8.7^3.11^2.13^2.17.19^2.23.29.31.37.41.131.593 \\
\end{align*}
\]
We shall explain the factors \( \ell = 373, 839, 2243 \) using congruences between Klingen-Eisenstein series and cusp forms of type \( \det^{14} \otimes \Sym^2(C^2) \), \( \det^{12} \otimes \Sym^4(C^2) \) and \( \det^{10} \otimes \Sym^6(C^2) \) respectively. The congruence \((\text{mod } 373)\) had already been proved by Satoh [Sa], following Kurokawa’s method (as we shall too). To produce explicitly the vector-valued Siegel modular forms that we need, we shall use a generalisation, due to Eholzer and Ibukiyama [EI], of the Rankin-Cohen brackets. In these cases we can take this elementary approach because the space of vector-valued cusp forms is 1-dimensional. However, to illustrate a different method (modelled on the proofs of Propositions 3.4 and 3.5 of [R]), by which most of the remaining cases can be handled, we prove a suitable congruence modulo \((a \text{ divisor of }) \ell = 9385577\). This is one of the large primes appearing in the table, below, of values of \( D_f(r + k' - 1)/(f, f)\pi^{r+k'-1} \) for odd \( r \) with \( 1 \leq r \leq k'-1 \) when \( k' = 20 \). This method is less elementary in that it depends on a formula of Böcherer, Satoh and Yamazaki for the Fourier coefficients of vector-valued Klingen-Eisenstein series [BSY]. For the most general statement about the congruences we may prove by this method, see Proposition 4.4. It covers all the large primes in numerators in the following table, and for \( k' = 22 \) above, only \( \ell = 59 \) and \( \ell = 23^2 \) are omitted.

In order to apply Urban’s construction, we need the 4-dimensional \( \lambda \)-adic Galois representation, attached to the vector-valued, genus 2 cusp form \( F \), to take values in \( \GL(4, K_\lambda) \), not just \( \GL(4, K_\lambda) \). This is expected to be true in general, and is certainly true in the \( k' = 16 \) examples where \( \dim S_{1, k} = 1 \). (See following Proposition 6.4 for further comment.)

\[
k' = 20
\]

\[
\begin{align*}
   r &= 1 & 2^{23} / 3^8.5.7.11.13.17.19 \\
         3    & 2^{25} / 3^{10}.5.7.11.13.17.19 \\
         5    & 2^{27} / 3^{12}.5.7.11.13.17.19.23 \\
         7    & 2^{29}.2593 / 3^{12}.5.7.11.13.17.19.23 \\
         9    & 2^{31}.8831 / 3^{15}.5.7.11.13.17.19.23 \\
        11    & 2^{37}.304793977 / 3^{19}.5.7.11.13.17.19.23.29 \\
        13    & 2^{39}.40706077 / 3^{20}.5.7.11.13.17.19.23.29.31 \\
        15    & 2^{41}.9385577 / 3^{19}.5.7.11.13.17.19.23.29.31 \\
        17    & 2^{43}.439367 / 3^{19}.5.11.13.17.19.23.29.31.37.283.617 \\
        19    & 2^{47}.7.11^2 / 3^{18}.5.7.11.13.17.19.23.29.31.37.283.617
\end{align*}
\]

It is interesting to compare with the situation for critical values of \( L_f(s) \). When \( k'/2 \) is odd, Brown [Br] has shown that, if \( \lambda \mid \ell \) is a large prime dividing the “algebraic part” of \( L_f(k'/2 + 1) \), then one can construct a non-zero element of \( \lambda \)-torsion in an appropriate Selmer group, using the Galois representation attached to a (non-CAP) scalar-valued Siegel cusp form of genus two, with Hecke eigenvalues congruent \((\text{mod } \lambda)\) to those of the Saito-Kurokawa lift of \( f \). To move away from this near-central critical value, one can try to use vector-valued Siegel cusp forms of genus two. Now there is no longer a Saito-Kurokawa lift, but Harder [Ha] has conjectured that cusp forms (of genus two) satisfying the necessary congruences exist anyway, and there is good computational evidence for his conjecture in special cases [FvdG], [vdG].

In §2 we introduce some definitions and notation. In §3 we describe (some of) Eholzer and Ibukiyama’s operators, and write down the Fourier expansions of some Siegel Eisenstein series. In §4 we use these ingredients to produce Fourier expansions
for some vector-valued Siegel modular forms, which are then used to prove the congruences modulo $839$ and $2243$ (for $k' = 16$), $9385577$ (for $k' = 20$) and more generally (Proposition 4.4). In §5 we examine the predictions of the Bloch-Kato conjecture, applied to the special case of critical twists of the symmetric square of the pre-motivic structure attached to $f$. The Galois representations attached to Siegel cusp forms are introduced in §6, and the construction of Urban is described. In §7 it is proved that this element of Galois cohomology satisfies the necessary local conditions to lie in the Bloch-Kato Selmer group. The main results on congruences and Selmer groups are summarised in Propositions 4.4 and 7.3 respectively.

In §8 we construct a non-zero Galois-fixed element in a certain $F_X[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module associated to the rightmost critical value of the standard $L$-function of the Siegel cusp form $F$. Then in §9 we show that, in favourable cases, $\lambda$ does appear in certain ratios of critical values, as predicted by the Bloch-Kato conjecture.

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2. **Klingen-Eisenstein series**

Let $\mathfrak{H}_2$ be the Siegel upper half plane of $2$ by $2$ complex symmetric matrices with positive-definite imaginary part. For $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_2 := \text{Sp}(4, \mathbb{Z})$ and $Z \in \mathfrak{H}_2$, let $M(Z) := (AZ + B)(CZ + D)^{-1}$ and $J(M, Z) := CZ + D$. Let $V$ be the space of the representation $\rho_{j,k} := \text{det}^k \otimes \text{Sym}^j(\mathbb{C}^2)$ of $\text{GL}(2, \mathbb{C})$. Let $\{u_1, u_1^{-1}u_2, \ldots, u_2\}$ be a standard basis for $V$. A holomorphic function $f : \mathfrak{H}_2 \to V$ is said to belong to the space $M_{j,k} = M_{j,k}(\Gamma_2)$ of Siegel modular forms of genus two and weight $\rho_{j,k}$ if

$$f(M(Z)) = \rho_{j,k}(J(M, Z))f(Z) \quad \forall M \in \Gamma_2, Z \in \mathfrak{H}_2.$$  

(We consider only even $j$, to avoid necessitating $f = 0$.) Such an $f$ has a Fourier expansion

$$f(Z) = \sum_{S \geq 0} a(S)e(\text{Tr}(SZ)) = \sum_{S \geq 0} a(S, f)e(\text{Tr}(SZ)),$$

where the sum is over all positive semi-definite matrices of the form $\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$ with $a, b, c \in \mathbb{Z}$, and $e(z) := e^{2\pi iz}$.

Let $\Gamma_1 = \text{SL}(2, \mathbb{Z})$ and $M_{k'}(\Gamma_1)$ the space of modular forms of weight $k'$ for $\Gamma_1$. The Siegel operator $\Phi$ on $M_{j,k}(\Gamma_2)$ is defined by

$$\Phi f(z) = \lim_{t \to \infty} f \left( \begin{bmatrix} z & 0 \\ 0 & \text{it} \end{bmatrix} \right) \quad \text{for} \quad z \in \mathfrak{H}_1, t \in \mathbb{R}.$$  

By Lemma 1.1 of [A], for $f \in M_{j,k}(\Gamma_2)$, $\Phi f$ is of the form $gu_1^j$ for some modular form $g \in M_{j+k}(\Gamma_1)$, in fact if $j > 0$ then $g \in S_{j+k}(\Gamma_1)$. We also denote $g = \Phi f$, so we have a linear operator $\Phi : M_{j,k}(\Gamma_2) \to M_{j+k}(\Gamma_1)$. In terms of Fourier expansions,

$$\Phi f(z) = \sum_{n \geq 0} a \left( \begin{bmatrix} n & 0 \\ 0 & 0 \end{bmatrix} \right) e(nz).$$  

(1)

The kernel of $\Phi$, denoted $S_{j,k}$, is the space of Siegel cusp forms of genus two and weight $\rho_{j,k}$. With respect to a natural inner product, there is an orthogonal decomposition $M_{j,k} = S_{j,k} \oplus N_{j,k}$, and, in the case $j > 0$, a linear isomorphism
with this in mind, given that the Andrianov (spinor) Eisenstein series to complete a basis for details. In the case \( P = 2 \) and \( S \), where \( S \) is the normalised Eisenstein series for \( \Gamma \), it has the property that \( \sum_{n=1}^{\infty} \sigma_{k-1}(n) e(nz) \) is the normalised Eisenstein series for \( \Gamma \). It may be obtained from a modular form of weight \( k - \frac{1}{2} \), via a Jacobi form of weight \( k \) and index 1, as described in §21 of [vdG]. This leads to a formula for its Fourier expansion:

\[
E_k^{(2)} = 1 - \frac{2k}{B_k} \sum_{S \geq 0} \left( \sum_{d|(a,b,c)} d^{k-1} H \left( k-1, \frac{4ac-b^2}{d^2} \right) \right) \text{e}(\text{Tr}(SZ)),
\]

where \( S = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \) and \( H(r,N) \) is the function defined by Cohen in [C]. Note that if \( D = (-1)^s N \) is the discriminant of a quadratic field then \( H(r,N) = L(1 - r, \chi_D) \). Using the tables of \( H(3,N) \) and \( H(5,N) \) at the end of [C], we obtain (abbreviating \( \text{e}(\text{Tr}(SZ)) \) to \( S \))

\[
E_4^{(2)} = 1 + 240 \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + 56 \begin{bmatrix} 1/2 & 1/2 \\ 1 & 1 \end{bmatrix} + \right) + 126 \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \right) + 2520 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \cdots
\]

and

\[
E_6^{(2)} = 1 - 504 \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - 88 \begin{bmatrix} 1/2 & 1/2 \\ 1 & 1 \end{bmatrix} \right) - 330 \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \right) - 49368 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \cdots.
\]

In later calculations we shall be especially concerned with the coefficients of \( P := \begin{bmatrix} 1/2 & 1/2 \\ 1 & 1 \end{bmatrix} \) and \( 2P = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \). The coefficients displayed above are chosen with this in mind, given that
\[
\begin{bmatrix}
1 & 1/2 \\
1/2 & 1
\end{bmatrix} + \begin{bmatrix}
1 & 1/2 \\
1/2 & 1
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix} + \begin{bmatrix}
2 & 1 \\
1 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} + \begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix}.
\]

It will be useful also to let \( J := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \).

In the next section we shall use these Siegel-Eisenstein series, together with Eholzer and Ibukiyama’s generalisation of the Rankin-Cohen bracket operators \([EI]\), to produce elements of \( M_{4,12} \) and \( M_{6,10} \) that will be useful to us. Note that by Tsushima’s formula \([Ts]\), \( S_{4,12} \) and \( S_{6,10} \) are both 1-dimensional. (See the table in §25 of \([vdG]\).)

For \( Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \), let \( R_i = \frac{\partial}{\partial z_i} \). Similarly, for \( Z' = \begin{bmatrix} z'_1 \\ z'_2 \end{bmatrix} \), let \( R'_i = \frac{\partial}{\partial z'_i} \).

For positive integers \( k_1, k_2 \) and \( v \), let \( D_{k_1,k_2,v} \) be the result of substituting \( S = R_1u_1^2 + R_2u_2^2 + 2R_3u_1u_2 \) and \( S' = R_1'u_1^2 + R_2'u_2^2 + 2R_3'u_1u_2 \) into the polynomial

\[
(2)
\sum_{r+s=v} (-1)^r \binom{v + k_2 - 1}{r} \binom{v + k_1 - 1}{s} S'^r S'^s.
\]

Now for \( F \in M_{0,k_1}(\Gamma_2), G \in M_{0,k_2}(\Gamma_2) \), let

\[
[F,G]_{2v} = D_{k_1,k_2,v}(F(Z)G(Z'))|_{z = z'}.
\]

Then by §5.3 and Proposition 6.1 of \([EI]\), \( [F,G]_{2v} \in M_{2v,k_1+k_2}(\Gamma_2) \). The bracket defined and used by Satoh in \([Sa]\) is a scalar multiple of \([.]_{2v} \). If we were to replace \( F \) and \( G \) by modular forms (for \( \Gamma_1 \)), and \( S, S' \) by \( \frac{\partial}{\partial z}, \frac{\partial}{\partial z'} \), we would obtain the Rankin-Cohen brackets of §7 of \([C]\).

The following is part of Theorem 4.1 of \([Sa]\). Let \( \eta_{14} \) be a generator of the 1-dimensional space \( S_{2,14} \), and for any Hecke eigenform \( f \in M_{j,k} \) let \( \nu(m,f) \) be the corresponding eigenvalue of \( T(m) \).

**Proposition 3.1** (Satoh). For all \( m \geq 1 \),

\[
\nu(m, \eta_{14}) \equiv \nu(m, [\Delta_{14}]_2) \quad \text{mod 373}.
\]

His proof follows the method employed by Kurokawa \([Ku]\) in the scalar case. We use the same method in the next section to prove some more congruences of similar type.

### 4. Some more congruences

The results of §§4.1 and 4.2 are subsumed by Proposition 4.4, but we include them for their more elementary proofs.

#### 4.1. \( \ell = 839 \)

Satoh used \( [E_6^{(2)}, E_8^{(2)}]_{12} \) and \( [E_6^{(2)}, E_{10}^{(2)}]_{12} \) to produce elements of \( M_{2,14} \). Similarly we can use \( [E_6^{(2)}, E_6^{(2)}]_4 \) and \( [E_8^{(2)}, E_4^{(2)}]_4 \) to produce elements of \( M_{4,12} \). The coefficient of \( u_1^4 \) in \( D_{4,4,2} \) is \( 10R_1^2 + 10R_1'^2 - 25R_1R_1' \). The coefficient of \( u_1^4 \) in \( D_{6,6,2} \) is \( 21R_1^2 + 21R_1'^2 - 49R_1R_1' \). Looking just at the terms for \( S = J, P \) and \( 2P \) (now \( S \) is a matrix again), we find that the \( u_1^4 \) part of \( [E_6^{(2)}, E_6^{(2)}]_4 \) is

\[
-504(42e(Tr(J))z - 3696e(Tr(PJ))z + 42311808e(Tr(2PJ))z + \ldots).
\]

We write \( [E_6^{(2)}, E_6^{(2)}]_4 =: -504\Phi_{12} \). It must be that \( \Phi_{12} = 42\Delta_{16} \), where \( \Delta_{16} = q + 216q^4 + \ldots \) is the normalised cusp form of weight 12 for \( \Gamma_1 \). (The coefficient
of \(e(\mathcal{L})\) turns into the coefficient of \(q = e(z)\), by (1). Note that \(\phi_{12}\) has integer Fourier coefficients.

We find also that the \(u_1^4\) part of \(E_4^{(2)}[E_4^{(2)}, E_4^{(2)}]_4\) is

\[
240 \left(20e(\text{Tr}(\mathcal{L})) + 1120e(\text{Tr}(\mathcal{P})) + 19872000e(\text{Tr}(2\mathcal{P})) + \ldots \right).
\]

Hence \(\Phi(42\left(1/240\right)E_4^{(2)}[E_4^{(2)}, E_4^{(2)}]_4 - 20\phi_{12}) = 0\). Dividing by \(120960 = 2^7 3^3 5 \cdot 7\), we get \(\eta_{12} \in S_{4, 12}\), where \(\eta_{12}\) has rational Fourier coefficients, integral at least for any \(\ell > 7\), and the \(u_1^4\) part is

\[
e(\text{Tr}(\mathcal{P})) - 96e(\text{Tr}(2\mathcal{P})) + \ldots.
\]

Since \(S_{4, 12}\) is 1-dimensional, \(\eta_{12}\) is necessarily a Hecke eigenform.

**Proposition 4.1.** For all \(m \geq 1\),

\[
\nu(m, \eta_{12}) \equiv \nu(m, [\Delta_{16}]_4) \pmod{839}.
\]

Note that, since \(\Delta_{16} \in S_{j + \ell}(\Gamma_1)\) with \(j = 4, k = 12\), so that \(k - 2 = 10\), we have, for any prime \(p\), \(\nu(p, [\Delta_{16}]_4) = (1 + p^{10})\nu(p, \Delta_{16})\).

**Proof.** Since \(\Phi[\Delta_{16}]_4 = \Delta_{16}, \Phi\phi_{12} = 42\Delta_{16}\) and \(\Phi\eta_{12} = 0\), and since \(\eta_{12}\) and \(\phi_{12}\) must span the 2-dimensional space \(\mathcal{M}_{4, 12}\), there must be a relation of the form

\[
[\Delta_{16}]_4 = \frac{1}{42} \phi_{12} + c\eta_{12}.
\]

Now

\[
a(2P, [\Delta_{16}]_4) = a(P, T(2)[\Delta_{16}]_4)
\]

(using (2.13) of [Sa])

\[
= \nu(2, [\Delta_{16}]_4) a(P, [\Delta_{16}]_4),
\]

and \(\nu(2, [\Delta_{16}]_4) = (1 + 2^{10})\nu(2, \Delta_{16}) = 1025 \cdot 216 = 221400\), hence

\[
a(2P, [\Delta_{16}]_4) = 221400e(P, [\Delta_{16}]_4).
\]

Substituting \(\Delta_{16} = \frac{1}{42} \phi_{12} + c\eta_{12}\) and looking at \(u_1^4\) parts, we obtain

\[
1007424 - 96c = 221400(-88 + c),
\]

hence 221496c = 20490624, so

\[
c = \frac{77616}{839}.
\]

Now applying \(T(m) - \nu(m, [\Delta_{16}]_4)\) to both sides of \(\Delta_{16} = \frac{1}{42} \phi_{12} + c\eta_{12}\), we see that the coefficients of \(\frac{77616}{839} \nu(m, \eta_{12}) - \nu(m, [\Delta_{16}]_4)\eta_{12}\) are integral at \(\ell = 839\), forcing the congruence to hold. (We use (2.13) of [Sa], and \(k \geq 2\), to see that \(T(m)\) preserves integrality of Fourier coefficients.)

Note that the coefficient 221496 of \(c\), which we divided by, is 221400 - (-96) = \(\nu(2, [\Delta_{16}]_4) - \nu(2, \eta_{12})\). The proof worked because this happened to be a multiple of 839, in other words we deduced the general case from the case \(m = 2\).
4.2. \(\ell = 2243\).

We can use \([E_4^{(2)} E_6^{(2)}]_6\) to produce an element of \(M_{6,10}\) with integer polynomials as Fourier coefficients. The coefficient of \(u_6^n\) in \(D_{4,6,3}\) is \(20R_1^2 - 120R_1R_2^2 + 168R_2^2R_1' - 56R_1\). Calculating as before, we find that \([E_4^{(2)} E_6^{(2)}]_6 = 5^3 \cdot 7 \cdot 7\phi_{10}\), where \(\phi_{10}\) has Fourier coefficients integral at any \(\ell > 7\), and its \(u_6^n\)-part is

\[
40\text{e}(\text{Tr}(PZ)) + 7342272\text{e}(\text{Tr}(2PZ)) + \ldots .
\]

Also \(\Phi\phi_{10} = -7\Delta_{16}\).

This time we cannot produce another element of \(M_{6,10}\), independent of \([E_4^{(2)} E_6^{(2)}]_6\), using brackets and Siegel-Eisenstein series. However, we do know that the space \(S_{6,10}\) is 1-dimensional. Proposition 5.1 of [BSY] shows that \([\Delta_{16}]_6\) has rational (polynomial) Fourier coefficients. We shall see below that it is not simply a multiple of \(\phi_{10}\). Therefore some linear combination of \([\Delta_{16}]_6\) and \(\phi_{10}\) gives a non-zero cusp form \(\eta_{10} \in S_{6,10}\) with rational polynomial Fourier coefficients. We choose \(\eta_{10}\) in such a way that some of its Fourier coefficients are integers with no non-trivial common divisor. Then for some integer \(a\), (the \(u_6^n\) part of)

\[
\eta_{10} = a\text{e}(\text{Tr}(PZ)) + \nu(2,\eta_{10})a\text{e}(\text{Tr}(2PZ)) + \ldots .
\]

G. van der Geer has kindly provided me with the eigenvalue \(\nu(2,\eta_{10}) = 1680\), calculated by the method described in [FvdG] and [vdG], which involves counting points on hyperelliptic curves over finite fields. [Strictly speaking, their method relies on an assumption (Conjecture 2 in §24 of [vdG]) about an “endoscopic contribution”, which appears to be closely related to (7) in Hypothesis A of [W].] Now copying the proof of the previous proposition, we find this time \(c = \frac{-1}{7\ell} (2134568 \cdot 2243)\). No matter what \(a\) is, we obtain then the following.

**Proposition 4.2.** For all \(m \geq 1\),

\[
\nu(m,\eta_{10}) \equiv \nu(m,[\Delta_{16}]_6) \pmod{2243}.
\]

Since \(\Delta_{16} \in S_{1+k}(\Gamma_1)\) with \(j = 6, k = 10\), so that \(k - 2 = 8\), this time we have, for any prime \(p\), \(\nu(p,[\Delta_{16}]_6) = (1 + p^8)\nu(p,\Delta_{16})\).

4.3. **Where the congruences come from.**

There is a slightly different approach to congruences such as the above, which illuminates their origin. In a relation such as

\[
[\Delta_{16}]_4 = \frac{1}{42} \phi_{12} + c\eta_{12},
\]

the important thing is to show that \(c\) has a factor of \(\ell\) (in this case 839) in the denominator. If it didn’t, this would imply that all the Fourier coefficients of \([f]_j\) are integral at \(\ell\). This is rendered extremely unlikely by Theorem 5.3 of [BSY], a formula for the Fourier coefficients (for non-singular \(S\)) of \([f]_j\), in which appears a multiple of \(D_\ell(2k' - 2 - j)\) in the denominator. Hence the appearance of \(\ell\) in the normalised \(D_\ell(2k' - 2 - j)\) explains its appearance in the denominator of \(c\), hence the existence of the congruence. This was noted at the end of §4 of [Sa], and was the motivation for [BSY]. Now we see what we can actually prove using this.

Let \(\Delta_{20}\) be the normalised cusp form of weight \(k' = 20\) for \(\Gamma_1\). Let \(\ell = 9385577\).

**Proposition 4.3.** There exists a Hecke eigenform \(\eta_{16} \in S_{4,16}\) such that, for all \(m \geq 1\),

\[
\nu(m,\eta_{16}) \equiv \nu(m,[\Delta_{20}]_4) \pmod{\lambda},
\]
where λ is a prime ideal dividing ℓ.

Note that, since Δ₂₀ ∈ S₁₉₊₄(Γ₁) with j = 4, k = 16, so that k − 2 = 14, we have, for any prime p, ν(p, [Δ₂₀|₄]) = (1 + p^1₄)ν(p, Δ₂₀).

**Proof.** Since dim[S₄,₁₆] = 3 > 1 (see §25 of [vdG]), the method used in previous examples does not apply.

Let c(S, u) (a polynomial in the entries of u = (u₁, u₂)) be the coefficient of e(Tr[S₂]) in [f]j, where f = Δ₂₀ = ∑ₙ₌₁∞ a(n)qⁿ and j = 4. In the case that det S is square-free, Theorem 5.3 of [BSY] tells us that

\[ H_{k,j}(S, u) = -2γ(k, j, 2)(\det S)^{(3/2) - k} \/ \alpha_{k,j} C_{k,j,1} D_f(2k' - 2 - j) \frac{H(k' - 1, \det S)}{(4π)^{k' - 1}} \sum_{x \in Z} a(S|x) \frac{Ω}{S| x |^{k' - 1}}, \]

where S[x] := xS^x x. In their theorem, our j is their l, p = 2, q = 1, n = 3. Their R is our S, and their outer sum has only one term, with T = S and ω₁ = 1₂₂. We must check that the factor of ℓ contributed by D_f(2k' - 2 - j)/f(Γ₂)π² + k' - 1 is not cancelled by anything else.

Up to powers of π, the constants αₖ,ₗ and Cₖ,₁,ₗ (defined on pages 15 and 9 respectively of [BSY]) are ℓ-units (using ℓ > 2k'), so may be disregarded. By Lemma 5.2 of [BSY], the generalised gamma factor H_{k,j}(S) almost cancels with the constant γ(k, j, 2).

In the case we shall concentrate on below, namely S = \( \begin{pmatrix} 1 & 1/2 & 1 \\ 1/2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \) (i.e. S = P), the factors of γ(k, j, 2)/H_{k,j}(S) (a power of det(S) = 3 and “P(S, S, S)/(u₄v)^{π₃}”, which may be evaluated using their (5.7)) are ℓ-units.

The inner sum \( \sum_{x \in Z} a(S|x) \frac{Ω}{S| x |^{k' - 1}} \) is the value at k' = 1 of the Rankin convolution of f and the weight-one theta series of S. Choosing S = \( \begin{pmatrix} 1 & 1/2 & 1 \\ 1/2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \), it becomes 6L(f, k' − 1)L(f, χ₃, k' − 1). Define Lalg(f, k' − 1) := L(f, k' − 1)/((2π)²)^{(k' − 1)Ω⁻} and Lalg(f, χ₃, k' − 1) := L(f, χ₃, k' − 1)/((2π)^{(k' − 1)Ω⁺}), where Ω⁺ and Ω⁻ are as in the next section. Observing, as in (4) in the next section, that the ratio of f to Ω⁺/Ω⁻ is an integer ±Ω₁, and that the Petersson norm of f may be cancelled with the one in D_f(2k' - 2 - j), we see that, to prove that ord_f(c(S, u)) < 0, it suffices to check that

\[ ℓ \not| H(k' - 1, 3)Lalg(f, k' - 1)Lalg(f, χ₃, k' - 1). \]

Using Stein’s table “Rational parts of the special values of the L-functions of level 1” [St], we find that ℓ ∤ Lalg(Δ₂₀, 1₉) (by which I mean ord_f(Lalg(Δ₂₀, 1₉)) ≤ 0), and using modular symbols as explained in §§8 and 2 of [MTT], we find that ℓ ∤ Lalg(Δ₂₀, χ₃, 1₉). Now H(1₉, 3) = L(1 − 1₉, χ₃) = −B₁₉₃/1₉ = \( \frac{7.19 \cdot 7691 \cdot 8629}{3} \). Certainly ℓ ∤ H(1₉, 3). We conclude that the Fourier coefficients of [Δ₂₀|₄] are not all integral at ℓ.


Let s be such that the Fourier coefficients of ℓʲ[Δ₂₀|₄] are integral at ℓ and not all divisible by ℓ. Then s > 0. With g as above, let \( h = ℓ^s[Δ₂₀|₄] − ℓ^s g \). Let
Let $S_{4,16}(\mathbb{Z}(\ell))$ denote the set of elements of $S_{4,16}$ with Fourier coefficients in $\mathbb{Z}(\ell)$. Then $h \in S_{4,16}(\mathbb{Z}(\ell))$, since $\Phi(g) = \Delta_{20}$. Furthermore, since the Fourier coefficients of $g$ are integral at $\ell$, the image of $h$ in $S_{4,16}(\mathbb{Z}(\ell))/\ell S_{4,16}(\mathbb{Z}(\ell))$ is a Hecke eigenform (with the same eigenvalues as $\Delta_{20}(\ell)$). (We use here that the Hecke operators preserve integrality of Fourier coefficients, which can be seen from (2.13) of [Sa].) The proof of the proposition is now completed by an application of the Deligne-Serre Lemma (Lemma 6.11 of [DeSc]).

Proposition 4.4. Let $f$ a normalised generator of $S_{k'}$, with $k' = 16, 18, 20, 22$ or 26. Suppose that $\ell > 2k'$ is a prime such that $\ell \mid D_f(r + k' - 1)/(f, f)\pi^{2r+k'-1}$, with $r$ odd and $1 \leq r < k'-1$. Suppose further that $r > 7$, if $k'/2$ is even, and $r \geq 9$ (but $r \neq 11$) if $k'/2$ is odd. Let $j = k'-1-r$ and $k = k'-j = r+1$. Then there exists a Hecke eigenform $F \in S_{1,k}$ such that, for all $m \geq 1$,

$$\nu(m, F) \equiv \nu(m, [f_j]) \pmod{\lambda},$$

where $\lambda$ is a prime ideal dividing $\ell$ in a sufficiently large field.

Proof. This can be done using exactly the same method as in Proposition 4.3 for $k' = 20, r = 15$. I have checked in all cases that $\ell$ does not divide (the numerator of) $L_{4,16}(f, k'-1), L_{4,16}(f, \chi, k'-1)$ or $H(k'-1, 3)$. Further, using a Maple program, I have checked that in none of the cases does $\ell$ appear in the factorisation of $P(S, S, S)/((u, v))^3$ (with $S = P$ again). The condition on $r$ is necessary in order to produce, using $E_4^{(2)}, E_6^{(2)}$ and brackets, a form with non-zero coefficient of $u_1^1 e(\text{Tr}(IZ))$ (which we can then divide by to get our $g$).

In a bit more detail, using (2) with $v = j/2$, the coefficient $c$ of $u_1^1 e(\text{Tr}(IZ))$ in $[E_4, E_4]_v$, is $1 + (-1)^v(\binom{v+3}{v}/v^3)$. In $[E_6, E_6]_v$ it is $1 + (-1)^v(\binom{v+5}{v}/v^3)$. For $v$ even these are non-zero and coprime to $\ell$, but for $v$ odd they are zero. When $v$ is odd, the coefficient $d$ of $u_1^1 e(\text{Tr}(IZ))$ in $[E_4, E_4]_v$ is $\binom{v+3}{v+2}...6 [(v+5)(v+4) - (5)(4)] = \frac{[(v+3)(v+2)...6]}{v}v(v+9)$, which is non-zero and coprime to $\ell$. When $v$ is even, it is $\frac{[(v+5)(v+4)+ (5)(4)] = \frac{[(v+3)(v+2)...6]}{v^2}}{v^2+9v+40}$, so we require the condition $\ell \mid (j^2+18j+160)$.

When $k'/2$ is even, for $k = 8$ we take $g = c^{-1}[E_4, E_4]_v$, with $v = (k'/2) - 4$, which is even. For $k = 10$ we take $g = d^{-1}[E_4, E_4]_v$, with $v = (k'/2) - 5$, which is odd. Each time $k$ goes up by 4, $v$ goes down by 2, and we multiply by a factor of $E_4$.

When $k'/2$ is odd, for $k = 14$ we take $g = c^{-1}[E_4, E_4]_v$, with $v = (k'/2) - 7$, which is even. For $k = 16$ we take $g = d^{-1}[E_4, E_4]_v$, with $v = (k'/2) - 8$, which is odd. Again, each time $k$ goes up by 4, $v$ goes down by 2, and we multiply by a factor of $E_4$. For $k = 10$ we take $g = d^{-1}[E_4, E_4]_v$, where $v = (k'/2) - 5$, which is even, so we require the condition $\ell \mid (j^2+18j+160)$, which certainly holds in the cases $k' = 18, 22, 26$.

See [Du] for the full table of values of $D_f(r + k' - 1)/(f, f)\pi^{2r+k'-1}$.

In the case $\dim S_{k'}(\Gamma_1) > 1$ it would be desirable to be able to apply the Deligne-Serre lemma directly to the reduction of $\lambda^* [f]$, which is a $(\text{mod } \lambda)$ cusp form (i.e. a section of some sheaf, as in [Hi2]), without having to produce an explicit cusp form like $h$. For this we would need to know that $(\text{mod } \lambda)$ cusp forms (of genus 2) can be lifted to characteristic-zero cusp forms.
5. The Bloch-Kato conjecture

Let \( f \in S_k((\Gamma_f) \) (necessarily for some even \( k' \geq 12 \)) be a normalised Hecke eigenform, \( K \) some number field containing all the Hecke eigenvalues of \( f \). Attached to \( f \) is a "premotivic structure" \( M_f \) over \( \mathbb{Q} \) with coefficients in \( K \). Thus there are 2-dimensional \( K \)-vector spaces \( M_{f, B} \) and \( M_{f, dR} \) (the Betti and de Rham realisations) and, for each finite prime \( \lambda \) of \( \mathcal{O}_K \), a 2-dimensional \( \mathcal{K}_\lambda \)-vector space \( M_{f, \lambda} \), the \( \lambda \)-adic realisation. These come with various structures and comparison isomorphisms, such as \( M_{f, B} \otimes K_\lambda \simeq M_{f, \lambda} \). See 1.1.1 of [DFG1] for the precise definition of a premotivic structure, and 1.6.2 of [DFG1] for the construction of \( M_f \). The \( \lambda \)-adic realisation \( M_{f, \lambda} \) comes with a continuous linear action of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \).

For each prime number \( p \), the restriction to \( \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \) may be used to define a local \( L \)-factor, and the Euler product is precisely \( L_f(s) \). Let \( M'_{f, \lambda} := \Sym^2 M_f \). Then similarly from \( M'_{f, \lambda} \) one obtains \( D_f(s) \).

On \( M_{f, B} \) there is an action of \( \text{Gal}(\mathbb{C}/\mathbb{R}) \), and the eigenspaces \( M^\pm_{f, B} \) are 1-dimensional. On \( M_{f, dR} \) there is a decreasing filtration, with \( F^j \) a 1-dimensional space precisely for \( 1 \leq j \leq k' - 1 \). The de Rham isomorphism \( M_{f, B} \otimes K \mathcal{C} \simeq M_{f, dR} \otimes K \mathcal{C} \) induces isomorphisms between \( M^\pm_{f, B} \otimes \mathcal{C} \) and \( (M_{f, dR}/F) \otimes \mathcal{C} \), where \( F := F^1 = \ldots = F^{k' - 1} \). Define \( \Omega^\pm \) to be the determinants of these isomorphisms. These depend on the choice of \( K \)-bases for \( M^\pm_{f, B} \) and \( M_{f, dR}/F \), so should be viewed as elements of \( \mathcal{C}^\times/K^\times \). Note that if we consider the twist \( M_f(j) \) (with \( 1 \leq j \leq k' - 1 \), then \( (M_f(j))_B = (2\pi i)^j M_f \), so \( (M_f(j))_B^+ = (2\pi i)^j M^+_f(j) \) and the Deligne period of \( M_f(j) \), as the determinant of the isomorphism from \( (M_f(j))_B^+ \otimes_K \mathcal{C} \) to \( (M_f(j))_B^{dR}/F \mathcal{C} \) induces an isomorphism between \( M^+_f \otimes \mathcal{C} \) and \( (M_f(j))_B^{dR}/F \mathcal{C} \), where \( F := F^{k'} = \ldots = F^{2k' - 2} \). Define \( \Omega \in \mathcal{C}^\times/K^\times \) to be the determinant of this isomorphism. Note that \( t \) as above is critical only when it is even, since this is when the dimension of \( (M_f(j))_B^+ = (2\pi i)^j M^+_f(j) \) matches that of \( (M_f(j))_B^{dR}/F \mathcal{C} \). In this case, the Deligne period of \( M_f(t) \) is \( (2\pi i)^{k' \Omega} \).

We shall choose an \( \mathcal{O}_K \)-submodule \( M_{f, B} \), generating \( M_{f, B} \) over \( K \), but not necessarily free, and likewise an \( \mathcal{O}_K[1/S] \)-submodule \( M_{f, dR} \), generating \( M_{f, dR} \) over \( K \), where \( S \) is the set of primes less than or equal to \( k' \). We take these as in 1.6.2 of [DFG1]. They are part of the "\( S \)-integral premotivic structure" \( M_f \) associated to \( f \). Actually, it will be convenient to enlarge \( S \) so that \( \mathcal{O}_K[1/S] \) is a principal ideal domain, then replace \( M_{f, B} \) and \( M_{f, dR} \) by their tensor products with the new \( \mathcal{O}_K[1/S] \). These will now be free, as will be any submodules and quotients. Choosing bases, and using these to calculate the above determinants, we pin down the values of \( \Omega^\pm \) (up to \( S \)-units). Setting \( M^+_{f, B} := \Sym^2 M_{f, B} \) and \( M^+_{f, dR} := \Sym^2 M_{f, dR} \), similarly we pin down \( \Omega \) (up to \( S \)-units). We just have to imagine not including in \( S \) any prime we care about (like 373, 839, 2243 or 9385577).

Lemma 5.1. \( \Omega = 2(2\pi i)^{1-k'} \Omega^+ \Omega^- \).

Proof. We follow Proposition 7.7 of [De]. Let \( e^+ \) and \( e^- \) be generators of \( M^+_{f, B} \) and \( M^+_{f, dR} \) respectively. Let \( \{x, y\} \) be an \( \mathcal{O}_K[1/S] \)-basis for \( M_{f, dR} \), with \( y \) generating the
Suppose that $t$ is a prime above $p$, so under the isomorphism between $M_{f,B}^t$ and $(M_{f,B}^t/F') \otimes \mathbb{C}$ we have

$$e^+ \mapsto \Omega^+ x + \eta^+ y, \quad e^- \mapsto \Omega^- x + \eta^- y,$$

for some $\eta^+, \eta^-$. Hence $\Omega = 2\Omega^+ \Omega^- \delta$, where $\delta = \Omega^\ast \eta^- - \Omega^- \eta^+$ is the determinant of the isomorphism $M_{f,B} \otimes \mathbb{K} \simeq M_{f,B} \otimes \mathbb{K}$ (with respect to the chosen $\mathcal{O}_K[1/S]$-bases).

We shall need the elements $\mathfrak{A}_\lambda$ of the S-integral pre-motivic structure, for each prime $\lambda$ of $\mathcal{O}_K$. These are as in 1.6.2 of [DFG1]. For each $\lambda$, $\mathfrak{M}_{\lambda}$ is a Gal($\Omega$/Q)-stable $\mathcal{O}_\lambda$-lattice in $M_{\lambda}$. Taking symmetric squares, we get $\mathfrak{M}_{\lambda}^2$, a Gal($\Omega$/Q)-stable $\mathcal{O}_\lambda$-lattice in $M_{\lambda}^2$.

Let $A_{\lambda} := M_{f,B}/\mathfrak{M}_{\lambda}$, and $A[\lambda] := A_{\lambda}/[\lambda]$ the $\lambda$-torsion subgroup. Similarly, let $A_{\lambda}^t := M_{t,\lambda}/\mathfrak{M}_{t,\lambda}$, and $A'[\lambda] := A_{\lambda}^t/[\lambda]$. Let $A_{\lambda}^t := M_{t,\lambda}/\mathfrak{M}_{t,\lambda}$, where $M_{t,\lambda}$ and $\mathfrak{M}_{t,\lambda}$ are the vector space and $\mathcal{O}_\lambda$-lattice dual to $M_{t,\lambda}$ and $\mathfrak{M}_{t,\lambda}$ respectively, with the natural Gal($\Omega$/Q)-action. Let $A' := \otimes A_{\lambda}'$, etc.

Following [BK] (Section 3), for $p \neq \ell$ (including $p = \infty$) let

$$H^1_t(Q_p, M_{f,\lambda}(t)) = \ker(H^1(Q_p, M_{t,\lambda}(t)) \to H^1(I_p, M_{t,\lambda}(t))).$$

Here $D_p$ is a decomposition subgroup at a prime above $p$, $I_p$ is the inertia subgroup, and $M_{t,\lambda}(t)$ is a Tate twist of $M_{t,\lambda}$, etc. The cohomology is for continuous cocycles and coboundaries. For $p = \ell$ let

$$H^1_t(Q, M_{t,\lambda}(t)) = \ker(H^1(D, M_{t,\lambda}(t)) \to H^1(D, M_{t,\lambda}(t) \otimes Q, B_{crys})).$$

(See Section 1 of [BK], or §2 of [Fo], for the definition of Fontaine’s ring $B_{crys}$.) Let $H^1_t(Q, M_{t,\lambda}(t))$ be the subspace of those elements of $H^1(Q, M_{t,\lambda}(t))$ that, for all primes $p$, have local restriction lying in $H^1_t(Q_p, M_{t,\lambda}(t))$. There is a natural exact sequence

$$0 \longrightarrow \mathfrak{M}_{t,\lambda}(t) \longrightarrow M_{t,\lambda}(t) \longrightarrow A_{\lambda}'(t) \longrightarrow 0.$$
We omit the definitions of the factors $c_p(t)$, but note that, since $M_f$ has “everywhere good reduction”, $c_p(t)$ is at worst a power of $p$ (or of $2$ if $p = \infty$). Furthermore, a direct application of Theorem 4.1(iii) of [BK] shows that $c_p(t)$ is trivial for all finite $p > 2^{k'} - 1$ (the length of the Hodge filtration of $M_f'$). Therefore if we are looking at the $\lambda$ part of the conjecture, with $\lambda \mid \ell > 2k'$, then these factors need not concern us. Note that $A'(\lambda)(k' - 1) \simeq \text{End}^0(A[\lambda])$ (endomorphisms of trace 0), and that $\text{Gal}(\mathbb{Q}/\mathbb{Q})$-invariants correspond to endomorphisms commuting with the $\text{Gal}(\mathbb{Q}/\mathbb{Q})$-action. Unless possibly if $\ell \leq k' + 1$ or $\ell \mid B_k$, $A[\lambda]$ is absolutely irreducible (by Lemma 8 of [SD]), and it follows from Schur’s lemma, in most cases, that the factors in the denominator have trivial $\lambda$ part. The exception is if $A[\lambda]$ is isomorphic to a non-trivial twist, but this can only happen if $\ell = 2k' - 1$ and $r = k' - 1$. (See the last section of [DH] for more details.)

It is more convenient to use $(f, f)$ than $\Omega$, so we consider the relation between the two. Bearing in mind §6 of [Hi1], using Lemma 5.1.6 of [De] and the latter part of 1.5.1 of [DFG1], one recovers the well-known fact that, up to $S$-units,

$$\langle f, f \rangle = t^{k' - 1} \Omega^+ \Omega^- c(f),$$

where $c(f)$, the “cohomology congruence ideal”, is, as the cup-product of basis elements for $M_{f,B}$, an integral ideal. Recall that by Lemma 5.1 above,

$$\Omega = 2(2\pi i)^{1 - k'} \Omega^+ \Omega^-.$$

Therefore (3) becomes, for $t = r + k' - 1$ with $1 \leq r \leq k'$ odd,

$$D_f(t + k' - 1) = \prod_{p < \infty} c_p(t) \# \Pi(t)\frac{\# H^0(Q, A'_{p}(t)) \# H^0(Q, A^c(1 - t)) c(f)}{\# H^0(Q, \lambda A'_{p}(1 - t)) c(f)}.$$  

Hence any large prime (e.g. $\ell = 373, 839, 2243$ or 9385577) dividing the left-hand side should be the order of an element in $\Pi(t + k' - 1)$. We shall construct such elements using the Galois interpretation of the congruences in §4, for $r = k' - 1 - 1$.

6. Galois representations

The following is part of Theorem I of [W]

Proposition 6.1 (Weissauer). Suppose that $\Pi$ is a unitary, irreducible, automorphic representation of $\text{GSp}(4, \mathbb{A}_\mathbb{Q})$ for which $\Pi_{\infty}$ belongs to the discrete series of weight $(k_1, k_2)$. Let $S$ denote the set of ramified places of the representation $\Pi$. Put $w = k_1 + k_2 - 3$. Then there exists a number field $E$ such that

1. for any prime $p \notin S$, if $L_p(p^{-s}) = L_p(\Pi_p, s - w/2)$ is the (shifted) local factor in the spinor $L$-function, then $L_p(1, \Pi_p) \in E[X]$;

2. for any prime $\lambda$ of $\mathcal{O}_E$, there exists a finite extension $K$ of $E$ (and $K_\lambda$ of $E_\lambda$), and a 4-dimensional semisimple Galois representation

$$\rho_{\Pi, \lambda} : \text{Gal}(\mathbb{Q}/\mathbb{Q}) \rightarrow \text{GL}(4, K_\lambda),$$

unramified outside $S \cup \{\ell\}$ (where $\lambda \mid \ell$), such that for each prime $p \notin S \cup \{\ell\}$,

$$L_p(\Pi_p, s - w/2) = \det(1 - \rho_{\Pi, \lambda}(\text{Frob}_p^{-1})p^{-s})^{-1}.$$  

The main theorems in [W] depend on hypotheses (A and B), whose proofs are currently only in preprint form. Section 3 of [U] gives an alternative proof of the above, in the case that $\ell \notin S$, $k_1 > k_2 > 3$ and the Newton polygon of $L_\ell(X)^{-1}$ has four distinct $\ell$-slopes.
Take a Hecke eigenform $F \in S_{1,k}$ (with $k \geq 3$). To $F$ may be associated a cuspidal automorphic form $\Phi_F \in L^2(\mathbb{Z}(A_Q)\text{GSp}(4,\mathbb{Q})\backslash\text{GSp}(4,\mathbb{A}_Q))$. This “lifting procedure” is described in detail in §3 of [AS] (§3.1 for the scalar-valued case, §3.5 for the vector-valued case). In the scalar-valued case one easily recovers $F$ from $\Phi_F$. I am grateful to R. Schmidt for explaining to me how this can be done also in the vector-valued case, even though $F$ is vector-valued and $\Phi_F$ is scalar-valued. The translates of $\Phi_F$ under a maximal compact subgroup $K_\infty \simeq U(2)$ of $\text{GSp}(4,\mathbb{R})$ generate a space of functions realising a representation of $U(2)$ isomorphic to the restriction of the contragredient of $\rho_{1,k}$. (This is the “lowest $K_\infty$-type” in a discrete series representation of $\text{GSp}(4,\mathbb{R})$.) One may assume that $\Phi_F$ itself corresponds to a highest-weight vector (by choosing the linear form $L$ in §3.5 of [AS] to be a highest weight vector). Let $V$ be the space of $\rho_{1,k}$ and $V^*$ the dual space. Given $\Phi_F$, one obtains a $V$-valued function $\Phi$ (from which the $V$-valued $F$ may be recovered) as follows. It suffices to specify the value (given $g \in \text{GSp}(4,\mathbb{A}_Q)$) of $\Phi(g)$ on any $f \in V^*$. But $f$ corresponds to some $K_\infty$-translate $\Phi_f$ of $\Phi_F$, and we may take $(\Phi_f(g))(f) = \Phi_f(g)$.

Let $\Pi_f$ be any irreducible constituent of the unitary representation of $\text{GSp}(4,\mathbb{A}_Q)$ generated by right translates of $\Phi_F$, as in §3.4 of [AS]. They are all isomorphic, in fact this unitary representation is expected to be irreducible already. In any case, we may choose $F$ in such a way that $\Phi_F$ generates an irreducible representation, and then $\Phi_F$ may be recovered from $\Pi_f$ as in Lemma 3.4.2 of [AS]. (In the vector-valued case, as explained to me by R. Schmidt, one must add, to the conditions in Lemma 3.4.2 of [AS], one about being killed by compact positive roots.) A consequence of the preceding discussion is the following, which will be used after Proposition 6.4.

**Lemma 6.2.** If the Hecke eigenspace of $F \in S_{1,k}$ is 1-dimensional, then the multiplicity of $\Pi_f$ in $L^2(\mathbb{Z}(A_Q)\text{GSp}(4,\mathbb{Q})\backslash\text{GSp}(4,\mathbb{A}_Q))$ is 1.

**Proposition 6.3.** Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_{k'}$ be a normalised Hecke eigenform. Let $\lambda | t > 2k'$ in a field $K$ large enough to contain the Hecke eigenvalues of $f$. Let $F \in S_{1,k}$ (with $j + k = k'$, $k \geq 4$ and $j \geq 0$ even) be a Hecke eigenform, and suppose that,

1. for all primes $p$,
   \[ \nu(p, F) \equiv \nu(p, [f]_j) = a_p(1 + p^{k-2}) \pmod{\lambda}; \]
2. $t \nmid B_{k'}$ (a Bernoulli number).

Then the following hold.

1. If $K$ is sufficiently large then there exists a 4-dimensional semisimple Galois representation
   \[ p_{F,\lambda} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}(4, K_\lambda), \]
   unramified outside $[t]$, such that for each prime $p \neq t$, $\det((1-p_{F,\lambda}(\text{Frob}_p^{-1})^{-1})p^{-s})^{-1}$ is the local factor in the (shifted) spinor $L$ function of $F$.
2. Choose a $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant $O_\lambda$-lattice $S$ in $W$ (the space of $p_{F,\lambda}$) and consider the representation $\overline{p}_{F,\lambda}$ of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $S/\Lambda S$. Then the composition factors of $\overline{p}_{F,\lambda}$ are $\overline{p}_{F,\lambda}$ and $\overline{p}_{F,\lambda}(2 - k)$.
3. $p_{F,\lambda}$ is absolutely irreducible.

**Proof.** (1) This is a direct consequence of Proposition 6.1, applied to $\Pi_f$. Note that here $k_1 = j + k, k_2 = k$, and the condition $k \geq 3$ (which, of course,
implies \( k \geq 4 \) is necessary to ensure that \( \Pi_{\infty} \) is discrete series. Also \( \Pi_F \) is unramified at all primes \( p \), since \( F \) is for the full modular group \( \Gamma_2 \).

(2) The congruence, with conclusion (1), implies that \( \text{tr}(\overline{\rho}_{F,\lambda}(\text{Frob}^{-1}_p)) = (1 + p^{k-2}) \text{tr}(\overline{\rho}_{F,\lambda}(\text{Frob}^{-1}_p)) \). It remains to observe that \( \overline{\rho}_{F,\lambda} \) is (absolutely) irreducible, a consequence of \( \ell > k' + 1 \) and \( \ell \not| B_{\ell^4} \) (by Lemma 8 of [SD]).

(3) If \( \overline{\rho}_{F,\lambda} \) were (absolutely) reducible, then its semisimplification would be on the list in the proof of Theorem 3.2.1 of [SU]. The fact that the composition factors of \( S/\lambda S \) are \( \overline{\rho}_{F,\lambda} \) and \( \overline{\rho}_{F,\lambda}(2-k) \) is incompatible with Cas A, and the condition \( \ell > k - 1 \) excludes Cas B (v), leaving only Cas B (iv), and hence implying that \( \pi_F \) is cuspidal associated to a Klingen parabolic. By Corollary 4.5 of [PS], this is impossible.

\[
\square
\]

**Proposition 6.4.** In the situation of Proposition 6.3, suppose further that there is a symplectic form on \( W \) with respect to which the image of \( \overline{\rho}_{F,\lambda} \) is contained in \( \text{GSp}(4, K_\lambda) \).

1. There is a choice of Galois invariant \( O_\lambda \)-lattice \( S \) in \( W \) such that \( B[\lambda] := S/\lambda S \) has a submodule, but not a quotient, realising the representation \( \overline{\rho}_{F,\lambda}(2-k) \).
2. Let

\[
\begin{align*}
0 \rightarrow A[\lambda][2-k] \xrightarrow{i} B[\lambda] \xrightarrow{\pi} A[\lambda] \rightarrow 0
\end{align*}
\]

be the exact sequence of \( F_\lambda[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})] \)-modules implied by (1). Let \( s \) be any section for \( \pi \) as a linear map of \( F_\lambda \)-vector spaces. Then there is an \( F_\lambda \)-valued skew-symmetric form \( (, ) \) on \( B[\lambda] \) such that

(a) each of \( i(A[\lambda][2-k]) \) and \( s(A[\lambda]) \) is an isotropic subspace;

(b) if \( [v_1, v_2] \) is a basis for \( A[\lambda][2-k] \) and \( [w_1, w_2] \) is the dual basis of \( A[\lambda] \) (relative to the natural duality \( A[\lambda] \times A[\lambda] \rightarrow F_\lambda(1-j-k) \)) then

\[
(s(w_1), i(v_1)) = \delta_{i,j}.
\]

Recall that the composition factors of \( B[\lambda] \) are isomorphic to \( A[\lambda] \) and \( A[\lambda][2-k] \). The proof of Proposition 6.4 is now a direct application of Theorem 1.1 and Proposition 1.1 of [U]. In our situation, condition (iv) of his Proposition 1.1 amounts to \( X_{\ell}^{2-k} \not\equiv X_{\ell}^{k-2} \pmod{\ell} \) (where \( X_{\ell} \) is the cyclotomic character), which holds because \( \ell - 1 > k - 2 \). The statements about isotropy are a consequence of this condition. The fact that \( \overline{\rho}_{F,\lambda}(2-k) \) cannot be both a quotient and a submodule of \( B[\lambda] \), i.e. that \( s \) cannot be a splitting of \( F_\lambda[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})] \)-modules, is a consequence of the absolute irreducibility of \( \overline{\rho}_{F,\lambda} \).

The condition that the image of \( \overline{\rho}_{F,\lambda} \) should be contained in \( \text{GSp}(4, K_\lambda) \), is expected to hold in general. See the remark following Theorem IV of [W] for how it might be possible to prove it. According to Theorem IV of [W], the condition does hold if the multiplicity of \( \Pi_F \) in \( L_2^2(Z(\mathbb{A}_Q)\text{GSp}(4, Q)\setminus\text{GSp}(4, \mathbb{A}_Q)) \) is 1, or even if \( \Pi_F \) is “weakly equivalent” to a representation for which the multiplicity is 1. Alternatively, Proposition 3.5 of [U] provides a proof (given the existence of \( \overline{\rho}_{F,\lambda} \)) that if the multiplicity of \( \Pi_F \) is 1 then the condition (that the image of \( \overline{\rho}_{F,\lambda} \) is contained in \( \text{GSp}(4, K_\lambda) \)) holds. Thanks to Lemma 6.2, we know that the condition will hold whenever the Hecke eigenspace of \( F \) is 1-dimensional. This includes, of course, those cases where \( \dim S_{1,k} = 1 \), i.e. \( k' = 16 \), \( \ell = 373, 839 \) or 2243. It also includes the scalar-valued cases \((k', \ell) = (20, 71), (22, 61), (22, 103)\) (where \( F \)
spans a complement to the Maass space of Saito-Kurokawa lifts, see §3 of [M]) and (26, 163), (26, 187273) (by the calculations in [Sk]). If the assumption (about the endoscopic contribution) underlying the calculations of [FvdG] is justified, then the condition also holds in the cases \( (j, k, \ell) = (2, 16, 541), (2, 16, 2879), (4, 14, 19501) \). In these examples \( \dim S_{j,k} = 2 \), and the eigenvalues of \( T(2) \) are distinct quadratic conjugates of each other, as described in §27 of [vdG].

Following Urban, in the situation of Proposition 6.4, we define a cocycle \( C : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Hom}(A[\lambda], A[\lambda](2 - k)) \) by

\[
C(g)(w) = i^{-1}(g.s(g^{-1}.w) - s(w)).
\]

This cocycle is not a coboundary, since \( s \) is not a splitting of \( \mathbb{F} \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \)-modules. A different choice of \( s \) would only change \( C(g) \) by a coboundary, so we have a well-defined, non-zero cohomology class \( [C] \in H^1(\mathbb{Q}, \text{End}(A[\lambda])(2 - k)) \).

**Lemma 6.5.** \([C] \in H^1(\mathbb{Q}, \text{End}^0(\mathbb{A}[\lambda](2 - k))) = H^1(\mathbb{Q}, \mathbb{A}[\lambda](j + 1))\).

**Proof.** Let \( C(g)(w_1) = av_1 + cv_2 \) and \( C(g)(w_2) = bv_1 + dv_2 \). Since \( s(A[\lambda]) \) is isotropic and \( g \) preserves the symplectic form (up to scalar multiples), we must have

\[
(s(w_1) + av_1 + cv_2, s(w_2) + bv_1 + dv_2) = 0,
\]

hence \( b - c = 0 \). Under the identification of \( A[\lambda] \) with \( A[\lambda](2 - k) \) obtained by just forgetting the twist, \( \{w_1, w_2\} \) corresponds to \( \{v_2, -v_1\} \) (if we scale the \( v_i \) appropriately), so the condition \( b - c = 0 \) is that the trace of the endomorphism \( C(g) \) of \( A[\lambda] \) is 0. Alternatively, using the duality to identify \( \text{Hom}(A[\lambda], A[\lambda])((2 - k)) \) with \( A[\lambda] \otimes A[\lambda](j + k - 1 + (2 - k)) = A[\lambda] \otimes A[\lambda](j + 1) \),

\[
C(g) = av_1 \otimes v_1 + cv_1 \otimes v_2 + bv_2 \otimes v_1 + dv_2 \otimes v_2,
\]

so the condition \( b = c \) tells us that \( C \) takes values in \( \text{Sym}^2(A[\lambda])(j + 1) = A[\lambda'](j + 1) \).

\[\Box\]

## 7. Checking local conditions

With assumptions as in Proposition 6.4, we have, following Urban, constructed a non-zero element \( [C] \in H^1(\mathbb{Q}, A[\lambda](j + 1)) \). As explained in §5, \( H^0(\mathbb{Q}, A[\lambda'](j + 1)) \) (and hence \( H^0(\mathbb{Q}, A[\lambda'](j + 1)) \)) is trivial, given that \( \ell > 2k \) and \( \ell \not\in \mathcal{B}_k \), so \( H^1(\mathbb{Q}, A[\lambda'](j + 1)) \) injects into \( H^1(\mathbb{Q}, A[\lambda](j + 1)) \), giving us a non-zero element \( c \). We would like to show that \( c \in H^1(\mathbb{Q}, A[\lambda](j + 1)) \), by showing that, for each prime \( p \), the restriction to a decomposition subgroup satisfies \( \text{res}_p(c) \in H^1_\mathfrak{p}(\mathbb{Q}_p, A[\lambda'](j + 1)) \).

**Lemma 7.1.** \( \text{res}_p(c) \in H^1_\mathfrak{p}(\mathbb{Q}_p, A[\lambda'](j + 1)) \) if \( p \neq \ell \).

**Proof.** When restricted to an inertia subgroup \( I_p \), the exact sequence

\[
0 \longrightarrow A[\lambda](2 - k) \longrightarrow B[\lambda] \longrightarrow A[\lambda] \longrightarrow 0.
\]

splits, since each term is a trivial \( I_p \)-module. Hence the image of \( [C] \) in \( H^1(I_p, A[\lambda](j + 1)) \) (and therefore of \( c \) in \( H^1(I_p, A[\lambda'](j + 1)) \)) is trivial. It follows easily, from this and the fact that \( A[\lambda'](j + 1) \) is unramified at \( p \), that \( \text{res}_p(c) \in H^1_\mathfrak{p}(\mathbb{Q}_p, A[\lambda'](j + 1)) \).

(See, for instance, Lemma 7.4 of [Br].)

Contrary to the suggestion at the end of [Du], it is straightforward to prove the local condition at \( p = \ell \). Note that Urban uses a different local condition at \( \ell \), and imposes an ordinarity condition. Recall that we are already assuming that
\( \ell > 2(j + k) \) and \( \ell \not| B_{1+k} \). To prove the local condition at \( p = \ell \), we need a somewhat stronger condition on \( \ell \).

**Lemma 7.2.** If \( \ell > 2j + 3k - 2 \) then \( \text{rest}(c) \in H^1_1(\mathbb{Q}_\ell, A^{\lambda}_j(j + 1)) \).

**Proof.** Both \( \rho_{\ell, \lambda} \) and \( \rho_{\ell, \lambda} \), restricted to \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \), are crystalline. (For a careful discussion of \( \rho_{\ell, \lambda} \), referring to [Fu], see 1.2.5 of [DFG1]. For \( \rho_{\ell, \lambda} \), see Theorem 3.2(ii) of [U], which refers to [Fu] and [CF].) It follows that there exist filtered Dieudonné \( \mathcal{O}_\lambda \)-modules \( \mathcal{D} \) and \( \mathcal{E} \) such that \( \mathcal{V}(\mathcal{D}) = \mathfrak{M}_{\ell, \lambda} \) and \( \mathcal{V}(\mathcal{E}) = S \) (with \( S \) as in the proof of Proposition 6.3). For this, and for the definitions of the modified Fontaine-Laffaille functor \( \mathcal{V} \) and the categories \( \mathcal{O}_\lambda \mathcal{M}^{\mathcal{F}\mathcal{a}} \) of filtered Dieudonné \( \mathcal{O}_\lambda \)-modules, see 1.1.2 of [DFG1]. We simply note that if \( \mathcal{M} \) is a filtered Dieudonné \( \mathcal{O}_\lambda \)-module (i.e., \( \mathcal{M} \in \mathcal{O}_\lambda \mathcal{M}^{\mathcal{F}\mathcal{a}} \)), and if \( \text{Fil}^{\ell}\mathcal{M} = \mathcal{M} \) and \( \text{Fil}^{\ell + 1}\mathcal{M} = \{0\} \), then \( \mathcal{M} \in \mathcal{O}_\lambda \mathcal{M}^{\mathcal{F}\mathcal{a}} \). Let \( 1_{\mathcal{F}\mathcal{D}} \) be such that \( \mathcal{V}(1_{\mathcal{F}\mathcal{D}}) = \mathcal{O}_\lambda \). Below, we will need to work in a category containing both the dual of \( \mathcal{D} \) (with graded pieces of degrees \( 1 - j - k \) and \( 0 \)) and \( \mathcal{D}[2 - k] \) (with graded pieces of degrees \( k - 2 \) and \( (k - 2) + (j + k - 1) \)), as well as \( 1_{\mathcal{F}\mathcal{D}} \). The condition on \( \ell \) ensures that \( \mathcal{O}_\lambda \mathcal{M}^{\mathcal{F}\mathcal{a}} \) is such a category. Let \( \mathcal{D} \) denote \( \mathcal{D}/\lambda \mathcal{D} \), etc. Note that in §4 of [BK], \( \mathcal{V} \) is “\( \mathcal{T} \).”

The exact sequence

\[
\begin{array}{cccccccccccc}
0 & \longrightarrow & A[\lambda](2 - k) & \overset{i}{\longrightarrow} & B[\lambda] & \overset{\pi}{\longrightarrow} & A[\lambda] & \longrightarrow & 0.
\end{array}
\]

is the image by \( \mathcal{V} \) of an exact sequence

\[
\begin{array}{cccccccccccc}
0 & \longrightarrow & \mathcal{D}[2 - k] & \overset{i}{\longrightarrow} & \mathcal{E} & \overset{\pi}{\longrightarrow} & \mathcal{D} & \longrightarrow & 0.
\end{array}
\]

Apply \( \text{Hom}_{\mathcal{O}_\lambda}(\mathcal{D}, -) \) to this sequence, then pull back by the inclusion of \( 1_{\mathcal{F}\mathcal{D}} \) in \( \text{Hom}_{\mathcal{O}_\lambda}(\mathcal{D}, \mathcal{D}) \) to get an element of \( \text{Ext}^1(1_{\mathcal{F}\mathcal{D}}, \text{Hom}_{\mathcal{O}_\lambda}(\mathcal{D}, \mathcal{D}[2 - k])) \). It is easy to check that this is sent by \( \mathcal{V} \) to the extension of \( \mathbb{F}_\lambda[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})] \)-modules corresponding to the class \( \text{rest}(\mathcal{C}) \).

Lemma 4.4 of [BK] gives a description of \( \text{Ext}^1(1_{\mathcal{F}\mathcal{D}}, \mathcal{M}) \) as a quotient of \( \mathcal{M} \), namely \( \mathcal{M}/(1 - \phi^0)(\text{Fil}^0\mathcal{M}) \). From this it is obvious that the natural map from \( \text{Ext}^1(1_{\mathcal{F}\mathcal{D}}, \text{Hom}_{\mathcal{O}_\lambda}(\mathcal{D}, \mathcal{D}[2 - k])) \) to \( \text{Ext}^1(1_{\mathcal{F}\mathcal{D}}, \text{Hom}_{\mathcal{O}_\lambda}(\mathcal{D}, \mathcal{D}[2 - k])) \) is surjective. Application of \( \mathcal{V} \) to any element of \( \text{Ext}^1(1_{\mathcal{F}\mathcal{D}}, \text{Hom}_{\mathcal{O}_\lambda}(\mathcal{D}, \mathcal{D}[2 - k])) \) produces an extension of \( \mathcal{O}_\lambda \) by \( \text{Hom}(\mathfrak{M}_{\ell, \lambda}, \mathfrak{M}_{\ell, \lambda}(2 - k)) \), an invariant \( \mathcal{O}_\lambda \)-lattice in a crystalline extension of \( K_{\lambda} \) by \( \text{Hom}(\mathfrak{M}_{\ell, \lambda}, \mathfrak{M}_{\ell, \lambda}(2 - k)) \). It is associated to a class in \( H^1_1(\mathbb{Q}_\ell, \text{Hom}(\mathfrak{M}_{\ell, \lambda}, \mathfrak{M}_{\ell, \lambda}(2 - k))) \) (the inverse image of \( H^1_1(\mathbb{Q}_\ell, \text{Hom}(\mathfrak{M}_{\ell, \lambda}, \mathfrak{M}_{\ell, \lambda}(2 - k))) \)). The subscript “\( \ell \)” is justified by the fact that the extension is crystalline.] Hence \( \text{rest}(\mathcal{C}) \in H^1_1(\mathbb{Q}_\ell, \text{Hom}(A[\lambda], A[\lambda](2 - k))) \) is in the image of \( H^1_1(\mathbb{Q}_\ell, \text{Hom}(\mathfrak{M}_{\ell, \lambda}, \mathfrak{M}_{\ell, \lambda}(2 - k))) \). It follows that \( \text{rest}(c) \in H^1_1(\mathbb{Q}_\ell, A^{\lambda}_j(j + 1)) \). \( \square \)

What we have actually proved may be summarised as follows.

**Proposition 7.3.** Let \( f = \sum_{n=1}^\infty a_n q^n \in S_k \) be a normalised Hecke eigenform. Let \( F \in S_{k, \ell} \) (with \( j + k = k' \), \( j \geq 0 \) even, \( k \geq 4 \)) be a Hecke eigenform, and suppose that, for all primes \( p \),

\[
\nu(p, F) \equiv \nu(p, f_{j'}) = a_p (1 + p^{k-2}) \quad (\text{mod} \lambda).
\]

Here, \( \lambda \mid \ell \geq 2j + 3k - 2 \) in a field \( k \) large enough to contain the Hecke eigenvalues of \( f \). Suppose also that \( \ell \not| B_{k'} \). Take \( k \) large enough for the representation \( \rho_{\ell, \lambda} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}(4, \mathbb{K}_{\lambda}) \)
in Proposition 6.3 to exist. Suppose that the image of $p_{F,\lambda}$ is contained in $\text{GSp}(4, K_\lambda)$. (See the discussion following Proposition 6.4.)

Then there exists a non-zero element in the Bloch-Kato Selmer group $H^1_1(Q, A'_\lambda(j + 1))$.

In the situation of Proposition 7.3, we have obtained a non-zero element $c \in H^1_1(Q, A'_\lambda(j + 1))$. Recall that $j$ is even. The point $s = j + 1$ is critical for $D_5(s)$ (i.e. the pre-motivic structure $M'_j(j + 1)$ is critical), and non-central, so conjecturally $H^1_1(Q, M'_{j,\lambda}(j + 1))$ is trivial. Admitting this, we have an element of $\lambda$-torsion in $\mathbb{III}(j + 1)$. The functional equation pairs $s = j + 1$ with $s = 2k' - 1 - (j + 1) = 2k' - 2 - j$. Then by [Fl] we get a non-zero element of $\lambda$-torsion in $\mathbb{III}(2k' - 2 - j)$, as required by the Bloch-Kato conjecture in the examples covered by Proposition 4.4. (Alternatively, just use the functional equation and look at $D_5(j + 1)$ instead.)

8. Global Torsion

We retain the notation of the previous three sections. There should exist a pre-motivic structure $M_F$ over $Q$ with coefficients in $K$ (if $K$ is chosen sufficiently large) such that $W$ is the $\lambda$-adic realisation $M_{F,\lambda}$. The $L$-function attached to $M'_F := \Lambda^2 M_F$ (equivalently to $\Lambda^2 W$) would be $L(M'_F, s) = \zeta(s - (2k + j - 3))L(F, s - (2k + j - 3), St)$, where $L(F, s, St)$ is the “standard $L$-function”. The right half of the set of critical points for $L(M'_F, s)$ should be $s = m + (2k + j - 3)$ with $m$ even and $0 < m \leq k - 2$. The Bloch-Kato conjecture reads (with $t := m + (2k + j - 3)$)

$$
\frac{L(M'_F, m + (2k + j - 3))}{(2\pi i)^{3m + (3k + j - 3)} - \Omega^\prime} = \frac{\prod_p c_p(t) \# \mathbb{III}'(t)}{\# H^0(Q, B'(1 - t))}.
$$

where the various terms are defined analogously to those in §5. It is really the $\lambda$ part of this formula that concerns us, so it suffices to say that we use the lattice $\Lambda^2 S$ in $\Lambda^2 W$, so that $B'[\lambda] = \Lambda^2 B[\lambda]$, with $S$ and $B[\lambda]$ as in §6.

Putting ourselves in the situation of §6, recall the exact sequence

$$
0 \longrightarrow A[\lambda](2 - k) \xrightarrow{i} B[\lambda] \xrightarrow{\pi} A[\lambda] \longrightarrow 0.
$$

of $\mathbb{F}_\lambda[\text{Gal}(\overline{Q}/Q)]$-modules. We see then that $B'[\lambda] = \Lambda^2 B[\lambda]$ has a 1-dimensional submodule isomorphic to $(\Lambda^2 A[\lambda](4 - 2k) \simeq \mathbb{F}_{\lambda}(1 - j - k)(4 - 2k) \simeq \mathbb{F}_{\lambda}(3 - j - 2k)(2 - k)$. Hence $H^0(Q, B'(t))$, with $t = (k - 2) + (2k + j - 3)$, has a non-trivial $\lambda$-part, which should contribute to the first term in the denominator of the right hand side of (6). Since $A[\lambda]$ is irreducible (as in the proof of Proposition 6.3), the only 1-dimensional composition factors of $B'[\lambda]$ are $\mathbb{F}_{\lambda}(5 - j - 3k)$ and $\mathbb{F}_{\lambda}(1 - j - k)$. It is easy to check that they contribute nothing to any of the other denominator terms for any critical value in the right half of the range. The $\lambda$-parts of all the $c_p(t)$ should be trivial (if $\ell > j + 3k - 3$), so short of the unlikely event that $\lambda$ appears in one of the $\# \mathbb{III}'(t)$ terms, the Bloch-Kato conjecture leads us to expect that for any even $m$ with $0 < m < k - 2$,

$$
\text{ord}_{\lambda} \left( \frac{\pi^{3m - (k - 2)}}{L(F, m, St)} \right) < 0.
$$

(We suppose also that $\ell \nmid B_{k-2}$, so we may disregard the $\zeta(k - 2)$ factor.)
9. The rightmost critical value

Proposition 9.1. Suppose that $S_{j+k}^1(\Gamma_1)$ is 1-dimensional, and let $\ell$ be a normalised generator, with $k \geq 3$. Let $k' = j + k$. Suppose that
\[
\text{ord}_\ell \left( D_\ell(2k' - 2 - j)/(f, f)\pi^{3k'-3-2} \right) = s > 0,
\]
and that $\ell > 2k + j$. Suppose also that the Fourier coefficients of the Klingen-Eisenstein series $[f]_j \in M_l,k(\Gamma_2)$ become integral at $\ell$ when multiplied by $\ell^a$, but not when multiplied by any smaller power of $\ell$. Take an even $m$ with $0 < m \leq k - 2$.

1. If $m = k - 2$ then there exists a Hecke eigenform $F \in S_{l,k}$, and a prime ideal $\mathfrak{P} \mid \ell$ in a suitable field, such that $\text{ord}_\mathfrak{P}(L(F, m, St)/\pi^{3m+2k+j-3}(F, F)) \leq -s$, with $F$ normalised in such a way that its Fourier coefficients are integral at $\lambda$ but not all divisible by $\lambda$.

2. If $m < k - 2$ and the Hecke eigenvalues of $F$ are not all congruent $(\mod \lambda)$ to those of some other Hecke eigenform in $S_{l,k}$, then
\[
\text{ord}_\lambda(L(F, m, St)/\pi^{3m+2k+j-3}(F, F)) \geq 0.
\]

Proof. (1) In our situation, the case $p = q = 2$ of Proposition 4.4 of [BSY] states that
\[
L^{(1)}E_k^{(4)}(Z, W) = \alpha_{k,j} \left[ C_{k,j,1} A(f) [f]_j(Z) [f]_j(W) + C_{k,j,2} \sum_T A(F) F(Z) F(W) \right],
\]
where the sum is over a basis of Hecke eigenforms in $S_{l,k}$, and the various terms will be discussed below. First, $A(f) = (\zeta(k)\zeta(2k-2)\zeta(2k-4))^{-1} D_\ell(2k' - 2 - j)/(f, f)$ and $A(F) = (\zeta(k)\zeta(2k-2)\zeta(2k-4))^{-1} L(k - 2, F, St)/(F, F)$. The constants $\alpha_{k,j}$ (defined in Remark 4.3 of [BSY]) and $C_{k,j,1}, C_{k,j,2}$ (defined in Proposition 3.1 of [BSY]) are powers of $\pi$ times explicit rational numbers involving factorials and products of small numbers. Given our assumption that $\ell$ is sufficiently large, the prime factorisations of these rational numbers do not involve $\ell$. As noted immediately before Theorem 2.2 of [Ka] (this is where we use $k \geq 3$), the Fourier coefficients of $(\zeta(k)\zeta(2k-2)\zeta(2k-4))E_k^{(4)}$ are integral at any prime greater than $2k - 1$. (The $\zeta$ factors are incorporated into his definition of the Eisenstein series, not the normalised one we use here.) The differential operator $L^{(1)}$ maps $M_{k, k}(\Gamma_2)$ to $M_{l,k}(\Gamma_2) \otimes M_{k,k}(\Gamma_2)$, and the formula (2.7) of [BSY] shows that it is integral at any prime greater than $k + j - 1$. Hence the modified left hand side $(\zeta(k)\zeta(2k-2)\zeta(2k-4))L^{(1)}E_k^{(4)}(Z, W)$ has Fourier coefficients integral at $\ell$. Since $[f]_j$ contributes twice to the first term on the right hand side, but $A(f)$ can only cancel one factor of $\ell^a$, some Fourier coefficient $c$ in the first term on the right hand side is such that $\text{ord}_\ell(c) = -s$ (ignoring powers of $\pi$). Considering the remaining sum on the right hand side, this is impossible unless $F$ as stated exists.

(2) For $m \neq k - 2$ there is (see (4.1) of [Koz]) a similar formula without the term involving $[f]_j$, with $A(F)$ involving $L(m, F, St)$, and $L^{(1)}D_{m+2}^{k-2-m}F_{m+2}^{(4)}(Z, W)$ on the left. The differential operator $D_{m+2}^{k-2-m}$ is as in [Bo], and using the definitions and formulas in that paper it is easy to see that it has the necessary integrality properties to show that again the (modified) left hand side will be integral at $\ell$, this time forcing $\text{ord}_\lambda(L(m, F, St)/\pi^{3m+2k+j-3}(F, F)) \geq$
Here \( F \) is the particular \( F \) appearing in (1), and the integrality of this one term is easily deduced from that of the sum, using the no-congruence condition. The absence of a term involving \([f]\) is due to the fact that \( \mathcal{D}_{m+2}^{k-2-m}E_{m+2}^{[4]}(Z,W) \) is a cusp form, by Satz 7 of [Bö].

\[ \square \]

In those cases with \( j + k = 16 \) and \( \ell = 373,839 \) or 2243, and also in the scalar valued cases \((k', \ell) = (20, 71), (22, 61), (22, 103), (26, 163) \) and \((26, 187273)\), the conditions of the proposition are certainly satisfied. In all these cases, the proposition confirms the prediction about the ratio of critical standard \( L \)-values, derived from the Bloch-Kato conjecture in the previous section.

Note that, as observed in the penultimate paragraph of §8 of [Du], there is an analogous phenomenon in the genus one case. Eisenstein primes in the denominators of rightmost normalised critical values of \( D_f(s) \) are accounted for by global torsion.

## References


UNIVERSITY OF SHEFFIELD, DEPARTMENT OF PURE MATHEMATICS, HICKS BUILDING, HOUNSFIELD ROAD, SHEFFIELD, S3 7RH, U.K.

E-mail address: n.p.dummigan@shef.ac.uk